**Solutions to Math 20E Midterm 2, Spring 2000, Lindblad.**

1. The unit normal to the surface $z = g(x,y) = x + y^2$ is given by $\mathbf{n} = (-g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k})/\sqrt{1+g_x^2+g_y^2} = (-\mathbf{i} - 2y\mathbf{j} + \mathbf{k})/\sqrt{2+4y^2}$ and $dS = \dfrac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} = \sqrt{1+g_x^2+g_y^2}\,dxdy = \sqrt{2+4y^2}dxdy$ so the area is

(a) $$\iint_S dS = \iint_D \sqrt{2+4y^2}dxdy = \int_0^1\int_0^y \sqrt{2+4y^2}dx\,dy.$$

(b) $$\int_0^1\int_0^y \sqrt{2+4y^2}dx\,dy = \int_0^1 y\sqrt{2+4y^2}dy = \frac{(2+4y^2)^{3/2}}{12}\Big|_0^1 = \frac{6^{3/2}-2^{3/2}}{12}$$

2. (a) Since $G(x,y,z) = x^2+y^2+z^2 = 1$ on the surface the unit normal is
$$\mathbf{n} = \frac{\nabla G}{|\nabla G|} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.$$

(b) The surface can be view as a graph $y = g(x,z) = \sqrt{1-x^2-z^2}$ over the region $D$ in the $xz$-plane defined by $y = \sqrt{1-x^2-z^2} \geq \sqrt{x^2+z^2}$, i.e. $D = \{(x,z);\, x^2+z^2 \leq 1/2\}$. Then $dS = \dfrac{dxdz}{|\mathbf{n}\cdot\mathbf{j}|} = \dfrac{dxdz}{y} = \dfrac{dxdz}{\sqrt{1-x^2-z^2}}$ and $\mathbf{F}\cdot\mathbf{n} = x + y = x + \sqrt{1-x^2-z^2}$ so

$$\iint_S \mathbf{F}\cdot\mathbf{n}dS = \iint_D \frac{x+\sqrt{1-x^2-z^2}}{\sqrt{1-x^2-z^2}}dxdz = \iint_D \Big(\frac{x}{\sqrt{1-x^2-z^2}}+1\Big)\,dxdz$$

If we introduce polar coordinates in the $xz$-plane, $x = r\cos\theta,\ z = r\sin\theta$, this becomes

$$\int_0^{1/\sqrt{2}}\int_0^{2\pi}\Big(\frac{r\cos\theta}{\sqrt{1-r^2}}+1\Big)\,d\theta r dr = \int_0^{1/\sqrt{2}}\Big(\frac{-r\sin\theta}{\sqrt{1-r^2}}+\theta\Big)\Big|_0^{2\pi} r dr = \int_0^{1/\sqrt{2}} 2\pi\, r dr = \pi r^2\Big|_0^{1/\sqrt{2}} = \frac{\pi}{2}$$

3. The region $R$ corresponds to the region $D = \{(u,v);\, 1/4 \leq u \leq 1/2,\ 1/4 \leq v \leq 1/2\}$. Now $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -e^{-y}\sin x & -e^{-y}\cos x \\ e^{-y}\cos x & -e^{-y}\sin x \end{vmatrix} = e^{-2y}(\cos^2 x + \sin^2 x) = e^{-2y}$ and
$\dfrac{\partial(x,y)}{\partial(u,v)} = \Big(\dfrac{\partial(u,v)}{\partial(x,y)}\Big)^{-1} = \dfrac{1}{e^{-2y}}$ so

$$\iint_R \frac{dxdy}{\cos^2 x} = \iint_D \frac{1}{\cos^2 x}\frac{\partial(x,y)}{\partial(u,v)}dudv = \int_{\frac14}^{\frac12}\int_{\frac14}^{\frac12}\frac{dudv}{e^{-2y}\cos^2 x} = \int_{\frac14}^{\frac12}\int_{\frac14}^{\frac12}\frac{du}{u^2}dv = -\frac{1}{4u}\Big|_{\frac14}^{\frac12} = \frac12$$

4. (a) The position vector is $\mathbf{R}(\theta,\phi) = \cos\theta(4+\cos\phi)\mathbf{i} + \sin\theta(4+\cos\phi)\mathbf{j} + \sin\phi\mathbf{k}$. Then $\mathbf{R}_\theta = -\sin\theta(4+\cos\phi)\mathbf{i} + \cos\theta(4+\cos\phi)\mathbf{j}$ and $\mathbf{R}_\phi = -\cos\theta\sin\phi\mathbf{i} - \sin\theta\sin\phi\mathbf{j} + \cos\phi\mathbf{k}$ so

$$\mathbf{R}_\theta\times\mathbf{R}_\phi = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta(4+\cos\phi) & \cos\theta(4+\cos\phi) & 0 \\ -\cos\theta\sin\phi & -\sin\theta\sin\phi & \cos\phi \end{bmatrix}$$
$$= (4+\cos\phi)\Big(\cos\phi(\cos\theta\mathbf{i}+\sin\theta\mathbf{j}) + \sin\phi\mathbf{k}\Big)$$

Hence $|\mathbf{R}_\theta\times\mathbf{R}_\phi|^2 = (4+\cos\phi)^2\big(\cos^2\phi\,(\cos^2\theta+\sin^2\theta)+\sin^2\phi\big) = (4+\cos\phi)^2$ so $dS = |\mathbf{R}_\theta\times\mathbf{R}_\phi|\,d\theta d\phi = (4+\cos\phi)\,d\theta d\phi$ and the area is

$$\iint_T dS = \int_0^{2\pi}\int_0^{2\pi}(4+\cos\phi)\,d\theta d\phi = 2\pi\int_0^{2\pi}(4+\cos\phi)\,d\phi = 2\pi\big(4\phi+\sin\phi\big)\Big|_0^{2\pi} = 16\pi^2$$

(b) The unit normal is $\mathbf{n} = \mathbf{R}_\theta\times\mathbf{R}_\phi/|\mathbf{R}_\theta\times\mathbf{R}_\phi| = \big(\cos\phi(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})+\sin\phi\mathbf{k}\big)$.