Math 20E Midterm 2, Winter 05, Lindblad.

1. (a) Let $\mathbf{F} = \mathbf{i} + \mathbf{j}$ and let $\mathbf{c}_1$ be the curve $\mathbf{c}_1 = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\,\mathbf{k}$, $0 \le t \le \pi/2$. Find the line integral $\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s}$.

(b) Let $\mathbf{F} = \mathbf{i} + \mathbf{j}$ and let $\mathbf{c}_2$ be the straight line segment from $(1,0,0)$ to $(0,2,\pi/2)$. Find the line integral $\int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$.

2. Find $\iint_R \frac{x-y}{(x+2y)^2}\,dxdy$, where $R = \{(x,y);\ 2 \le x+2y \le 4,\ 0 \le x-y \le 3\}$, by making a change of variables.

3. Let S is the closed surface of the region $W = \{(x,y,z);\ x^2+y^2+2 \le z \le 6\}$, i.e. S is the surface consisting of the two parts S_1 and S_2, where $S_1 = \{(x,y,z);\ z = x^2+y^2+2,\ x^2+y^2 \le 4\}$, $S_2 = \{(x,y,z);\ z = 6,\ x^2+y^2 \le 4\}$.

a) Find the area of S.

b) Find the flux of $\mathbf{F} = -\mathbf{k} + x\mathbf{i}$ out through S; $\iint_S \mathbf{F} \cdot \mathbf{n}\,dS$. Here $\mathbf{n}$ is the unit normal oriented out from W.

4. Let S be the surface given by the parametrization

$$\mathbf{T}(u,v) = v\,\cos u\,\mathbf{i} + v\,\sin u\,\mathbf{j} + (1+v)\,\mathbf{k}, \qquad 0 \le u \le 2\pi, \quad 1 \le v \le 2.$$

a) Find the area element dS expressed in terms of the parametrization $du\,dv$.

b) Find the surface area using the parametrization above.

Math 20E Midterm 2 Solutions.
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1.a. Define $F(x,y,z) := (1,1,0)$,
and define $c_1(t) := (\cos(t), 2\sin(t), t)$,
for $t \in [0, \pi/2]$. Find $\int_{c_1} F \cdot ds$.

Solution:

$$F(c_1(t)) = F(\cos(t), 2\sin(t), t)$$
$$= (1,1,0).$$

$$c_1'(t) = (-\sin(t), 2\cos(t), 1).$$

$$F(c_1(t)) \cdot c_1'(t) = (-\sin(t) + 2\cos(t))$$

Thus

$$\int_{c_1} F \cdot ds = \int_0^{\pi/2} F(c_1(t))\, c_1'(t)\, dt$$
$$= \int_0^{\pi/2} (-\sin(t) + 2\cos(t))\, dt$$

$$= \left[\cos(t)\right]_0^{\pi/2} + 2\left[\sin(t)\right]_0^{\pi/2}$$

$$= \left[0 - 1\right] + 2\left[1 - 0\right]$$

$$= 1.$$

1.b. Define $F(x, y, z) := (1, 1, 0)$,
and define $c_2(t) := \big(((0, 2, \pi/2) - (1, 0, 0))t$
$+ (1, 0, 0)\big)$

$= (-t + 1, 2t, \pi/2\, t)$, for $t \in [0, 1]$.

Then c_2 parameterizes a straight line from $(1, 0, 0)$ to $(0, 2, \pi/2)$.

Find $\int_{c_2} F \cdot ds$

Solution:

$$F(c_2(t)) = F((1-t, 2t, \pi/2\, t))$$
$$= (1, 1, 0)$$

$c_2'(t) = (-1,\ 2,\ \pi/2).$

$$F(c_2(t)).c_2'(t) = (1,1,0).(-1,2,\pi/2)$$
$$= (-1)+2$$
$$= 1.$$

Thus

$$\int_{c_2} F.ds = \int_0^1 F(c_2(t)).c'(t)dt$$
$$= \int_0^1 1.dt$$
$$= 1.$$

Notice that the two curves c_1 and c_2 have the same start and end point.

2. Define $R = \{ (x,y) : 2 \leq x + 2y \leq 4,\ 0 \leq x - y \leq 3 \}$. Find

$$\iint_R \frac{x-y}{(x+2y)} \, dx\,dy.$$

Solution: Let $u = x + 2y$ and let $v = x - y$. Then we have

$$\frac{x-y}{(x+2y)^2} = \frac{v}{u^2}$$

$$dx\,dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

or

$$du\,dv = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dx\,dy$$

$$= \left| \begin{matrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{matrix} \right| dx\,dy$$

$$= \left| \det \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \right| dxdy$$

$$= |(-1-2)|\,dxdy$$

$$= 3\,dxdy.$$

Thus $dxdy = \frac{1}{3} du dv$. Thus

$$\iint_R \frac{x-y}{(x+2y]} dx\,dy$$

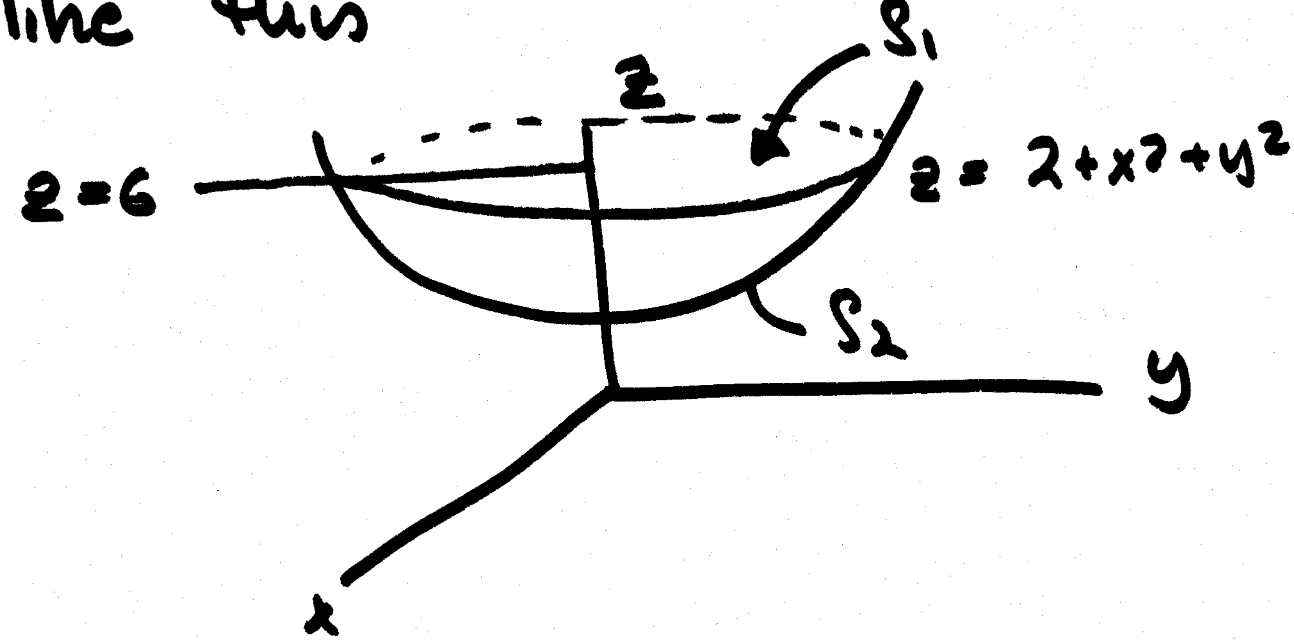
$$= \int_0^3 \int_2^4 \frac{v}{u^2} \frac{1}{3} du dv$$

$$= \frac{1}{3} \left[\frac{v^2}{2} \right]_0^3 \left[\frac{(-1)}{u} \right]_2^4$$

$$= \frac{1}{3} \cdot \frac{9}{2} \cdot \left[-\frac{1}{4} - -\frac{1}{2} \right] = \frac{1}{3} \cdot \frac{9}{2} \cdot \frac{1}{4} = \frac{3}{8}$$

3.a. Let $S := \{ (x, y, z) : x^2 + y^2 + 2 \leq z \leq 6 \}$. Find the area of S.

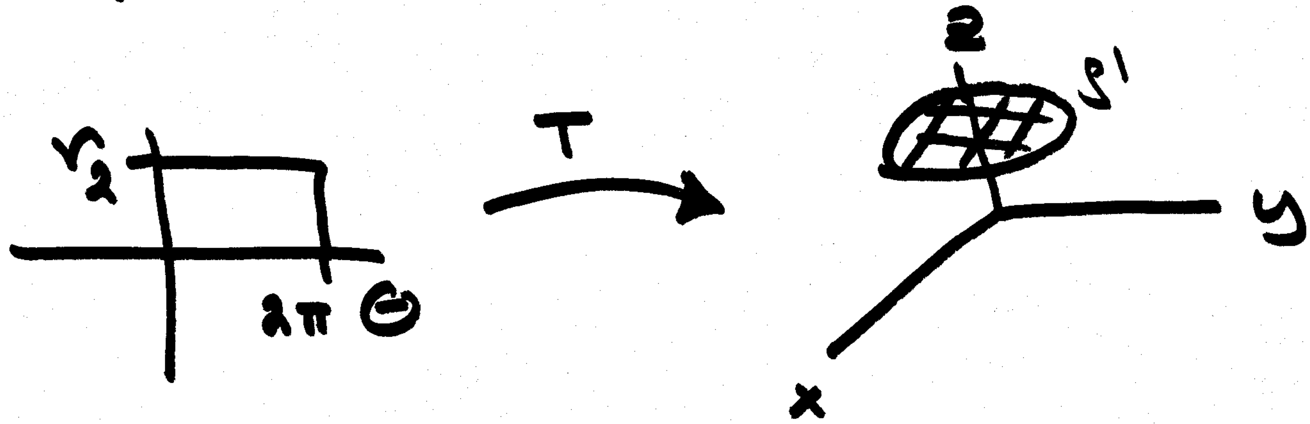
Solution: the surface S looks like this



Remember that the top, S_1, is also included in the surface. Thus

$$\int_S 1\, dS = \int_{S_1} 1\, dS + \int_{S_2} 1\, dS$$

Let us begin with S_1. Here is a possible parameterization of S_1.



$$(r, \theta) \xmapsto{\ T\ } (r\cos\theta, r\sin\theta, 6)$$

Thus

$$(\partial T/\partial \theta)(r, \theta) = (-r\sin\theta, r\cos\theta, 0)$$
$$(\partial T/\partial r)(r, \theta) = (\cos\theta, \sin\theta, 0)$$

Thus

$$(\partial T/\partial \theta \wedge \partial T/\partial r)(\theta, r) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix}$$

$$= \hat{i}[0] + \hat{j}[0] + \hat{k}[-r]$$

Thus

$$\left|\partial T/\partial\theta \wedge \partial T/\partial r\right|(r,\theta) = r$$

Hence,

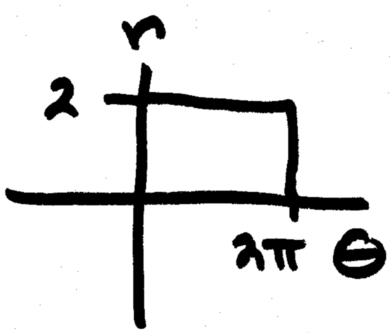
$$\int_{S_1} 1 \, dS = \int_0^{2\pi}\int_0^2 r\,dr\,d\theta$$

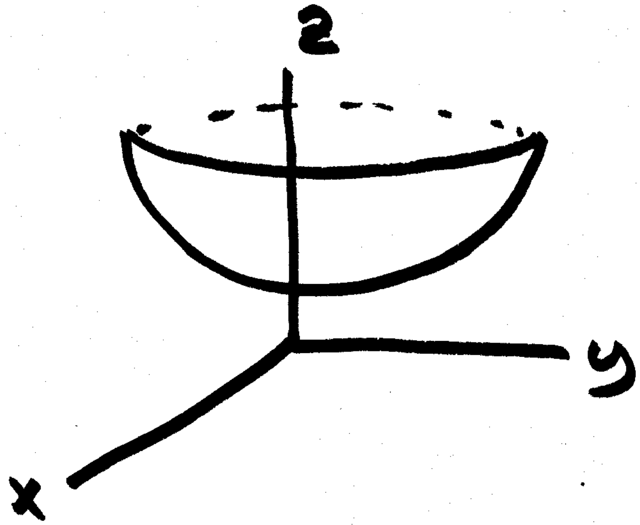
$$= \left[\frac{r^2}{2}\right]_0^2 2\pi$$

$$= 4\pi.$$

(this could also have been obtained much more simply... How?)

Now we parameterize S_2.





$$(r,\theta) \xmapsto{\;T\;} (r\cos\theta,\ r\sin\theta,\ 2+r^2)$$

Thus

$$(\partial T/\partial r)(r,\theta) = (\cos\theta,\ \sin\theta,\ 2r)$$

$$(\partial T/\partial \theta)(r,\theta) = (-r\sin\theta,\ r\cos\theta,\ 0)$$

Thus

$$(\partial T/\partial r \wedge \partial T/\partial \theta)(r,\theta) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= -\hat{i}\,[2r^2\cos\theta]$$
$$-\hat{j}\,[2r^2\sin\theta]$$
$$+\hat{k}\,[+r]$$

$$= (-2r^2\cos\theta,\ -2r^2\sin\theta,\ +r)$$

Thus

$$|\partial T/\partial r \wedge \partial T/\partial \theta|(r,\theta) = \sqrt{4r^4 + r^2}$$

Thus

$$\int_{S_2} 1\, dS = \int_0^{2\pi}\int_0^{2} 1 \cdot \sqrt{r^2(4r^2+1)}\, dr\, d\theta$$

$$= 2\pi \int_0^{2} r\sqrt{1+4r^2}\, dr$$

Let $u = 1+4r^2$. Then $du = 8r\,dr$.

Thus

$$= 2\pi \int_1^{9} \sqrt{u}\ \frac{du}{8}$$

$$= \frac{2\pi}{8}\left[\frac{2u^{3/2}}{3}\right]_1^9$$

$$= \frac{4\pi}{24}\left[9^{3/2} - 1\right]$$

$$= \frac{\pi}{6}\left[27 - 1\right] = \frac{26\pi}{6}$$

$$= \frac{13\pi}{3}$$

Combine the results for the final answer.

3. b. Define $F(x, y, z) := (x, 0, -1)$.
Find $\int_S F \cdot dS$.

Solution: Let us again begin by considering S_1, with the same parameterization. Then we have

$$\vec{F}(r, \theta) := (F \circ T)(r, \theta)$$
$$= F(r\cos\theta, r\sin\theta, 6)$$
$$= (r\cos\theta, 0, -1)$$

Thus

$$\int_{S_1} F \cdot dS = \int\int_{S'} (r\cos\theta, 0, -1) \cdot (0, 0, -r)$$

where did this come from?

$$= \int_0^{2\pi}\int_0^2 -r\,dr\,d\theta$$

$$= -4\pi.$$

Now lets consider S_2. Then

$$\vec{F}(r,\theta) = (F \circ T)(r,\theta)$$

$$= (r\cos\theta,\ 0,\ -1)$$

and

$$\int_{S_2} \vec{F}\,dS = \int\int_{S_2} (r\cos\theta, 0, -1) \cdot (+2r^2\cos\theta, +2r^2\sin\theta, -r)\,dr\,d\theta$$

$$= \int\int (+2r^3\cos^2\theta + r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 + 2r^3 \cos^2\theta$$

$$+ 2\pi \int_0^2 r\,dr$$

$$= 4\pi + \left[+ \frac{r^4}{2} \right]_0^2 \int_0^{2\pi} \frac{1}{2}(\cos 2\theta + 1)\,d\theta$$

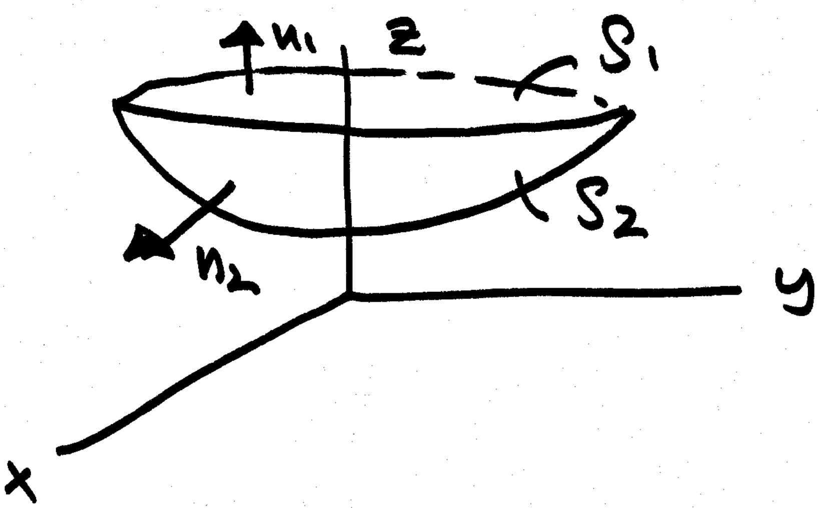
$$= 4\pi + \left[\frac{-2^4}{2} \right] \cdot \left[\frac{1}{2}\cdot\frac{1}{2}\sin 2\theta + \frac{\theta}{2} \right]_0^{2\pi}$$

$$= 4\pi + (+8)\cdot\pi$$

$$= 12\pi$$

Thus $\int_S F \cdot dS = 8\pi$

Alternative solution to question 3:
The surface looks like this



Note... I labelled the surfaces backwards

$F(x, y, z) := (x, 0, -1)$.

Let us first find $\int_{S_1} F \cdot dS$.

That is, we must find

$$\int\int_R F(x, y, z) \cdot n_1 \frac{dx\,dy}{|n \cdot k|},$$

where R is the region in the xy-plane onto which we project S_1.

$$F(x, y, z) = F(x, y, 6)$$
$$= (x, 0, -1).$$

$n_1 = \hat{k}$. Thus

$F(x,y,z) \cdot n_1 = -1$, and $|n_1 \cdot \hat{k}| = 1$.

Thus

$$\int_{S_1} F \cdot dS = \int\int_R -1 \, dx dy$$

Switch to polars

$$= -\int_0^2 \int_0^{2\pi} r dr d\theta$$

$$= -2\pi \left[\frac{r^2}{2} \right]_0^2$$

$$= -2\pi \frac{4}{2} = -4\pi.$$

Let us now consider S_2. S_2 is given by

$$\left\{ (x,y,z) : \begin{array}{c} z = x^2 + y^2 + 2 \\ x^2 + y^2 \leq 4 \end{array} \right\}.$$

Thus

$$n_2 = \nabla(z - x^2 - y^2)$$
$$= (-2x, -2y, 1).$$

Note that this normal vector points into ω. Thus we want $n_2 = (2x, 2y, -1)$.

$$F(x, y, z) = F(x, y, 2 + x^2 + y^2)$$
$$= (x, 0, -1).$$

Thus

$$F(x, y, z) \cdot n_2 = (x).(2x) + (0).(2y) + (-1)(-1)$$
$$= 2x^2 + 1.$$

and

$$|n_2 \cdot \hat{k}| = 1$$

Thus

$$\int_{S_2} F\,dS = \iint_R (2x2+1)\frac{dxdy}{1},$$

where R is the region in the xy-plane onto which we project S_2. Switching to polars

$$= \int_0^2 \int_0^{2\pi} (2r^2\cos^2\theta + 1)\,r\,dr\,d\theta$$

$$= 2\left[\frac{r^4}{4}\right]_0^2 \int_0^{2\pi} \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$+ \left[+\frac{r^2}{2}\right]_0^2 \int_0^{2\pi} d\theta$$

$$= \frac{2^5}{2^3}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$+ \frac{2^2}{2}\cdot 2\pi$$

$$= 2^2\cdot 2\pi + 2^2\pi$$

$$= 12\pi$$

Thus $\int_{S_1 \cup S_2} F\,dS = 8\pi$

4.a. Let S be the surface parameterized by $T: [0, 2\pi] \times [1, 2] \to S$, where

$T(u,v) := (v\cos(u), v\sin(u), (1+v))$

Find dS.

Solution:

$$(\partial T/\partial u)(u,v) = (-v\sin(u), v\cos(u), 0)$$

$$(\partial T/\partial v)(u,v) = (\cos(u), \sin(u), 1)$$

Then

$$(\partial T/\partial u \wedge \partial T/\partial v)(u,v) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix}$$

$$= \hat{\imath}[v\cos(u)] + \hat{\jmath}[v\sin(u)] + \hat{k}[-v]$$

Then

$$\left| \partial T/\partial u \times \partial T/\partial v \right| (u,v) = \sqrt{v^2 \cos^2(u) + v^2 \sin^2(u) + v^2}$$

$$= \sqrt{2}\, v$$

Thus $dS = \sqrt{2}\, v \, du \, dv$.

4. b. Find the area of S.

Solution:

$$\text{Area of } S = \iint_S 1 \, dS$$

$$= \int_0^{2\pi} \int_1^2 v \sqrt{2} \, dv \, du$$

$$= \sqrt{2} \cdot 2\pi \cdot \left[\frac{v^2}{2} \right]_1^2$$

$$= \frac{\sqrt{2} \cdot 2\pi}{2} \left[4 - 1 \right]$$

$$= 3 \cdot \sqrt{2} \cdot \pi$$