Math 20F Final Spring 04, June 8. Lindblad.

1. Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \\ -1 & -1 & -1 \end{pmatrix}$

(a) Find the inverse of A. (20p)
(b) Let $\mathbf{b} = (1,0,0)^T$. Use (a) to solve $A\mathbf{x} = \mathbf{b}$. (5p)

2. Let $A = \begin{pmatrix} 1 & 2 & 3 & -4 & 5 \\ -1 & -1 & -4 & 3 & 2 \\ 2 & 4 & 6 & 6 & -4 \end{pmatrix}$

(a) Find a basis for the nullspace of A, (10p)
(b) Find a basis for the row space of A, (10p)
(c) Find a basis for the column space of A. (10p)

3. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

(a) Show that $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are linearly independent. (10p)
(b) Determine if $\mathbf{b}$ is in the span of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$. (10p)

4. Let F be the plane $2x_1 + x_2 - x_3 = 0$.
(a) Find an orthonormal basis for F. (Hint: Start by picking any basis for F.) (10p)
(b) Find the matrix which describes the orthogonal projection from $\mathbf{R}^3$ onto F. (15p)

5. Let L be the linear transformation from $\mathbf{R}^2 \to \mathbf{R}^2$ given by rotating counterclockwise around the origin by an angle $\theta = \pi/2$, followed by reflection in the x_2 axis.
(a) Find the matrix representative A of L with respect to the standard basis for $\mathbf{R}^2$. (Hint: Start by finding the matrix for the rotation.) (10p)
(b) L has two eigenvectors $\mathbf{u}_1 = (1,1)^T$, $\mathbf{u}_2 = (1,-1)^T$. Find their eigenvalues λ_1, λ_2(5p)
(c) Find the matrix for L with respect to $\mathbf{u}_1$ and $\mathbf{u}_2$ in (b). (5p)

6. (a) Find the line $y = ax + b$ which best fits the points
$(x, y) = (-2,-3)$, $(-1,-2)$, $(0,2)$, $(1,6)$, $(2,7)$, in the least squares sense. (15)
(b) Set up the least square problem to find the quadratic polynomial $y = a + bx + cx^2$ that best fits the points. (do not solve it) (5p)

7. For each of the matrices below, either diagonalize it, i.e. find a matrix X such that $X^{-1}AX$ etc. is diagonal, or explain why its not possible: (10p each)

$$(a) \quad A = \begin{pmatrix} 4 & 0 \\ 2 & 5 \end{pmatrix}, \qquad (b) \quad B = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \qquad (c) \quad C = \begin{pmatrix} 16 & -12 & 0 \\ -12 & 9 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

8. (a) Calculate e^{At}, where $A = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$. (20p)
(b) Solve the initial value problem (10p)

$$\begin{aligned} y_1' &= 4y_1 + 2y_2, & y_1(0) &= 1 \\ y_2' &= -y_1 + y_2, & y_2(0) &= 2. \end{aligned}$$