Lecture 10: 4.2 Null space and Column space..
The **null space** of an $m \times n$ matrix A is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$; Nul $A = \{\mathbf{x} \in \mathbf{R}^n;\ A\mathbf{x} = \mathbf{0}\}$.
Theorem. The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$.
Proof. We must verify the three properties (a), (b), (c) in the definition of subspace.
(a) $\mathbf{0} \in$ Nul A since $A\mathbf{0} = \mathbf{0}$.
(b) If $\mathbf{u}, \mathbf{v} \in$ Nul A, show that $\mathbf{u} + \mathbf{v} \in$ Nul A. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.
(c) If $\mathbf{u} \in$ Nul A, show that $\lambda\mathbf{u} \in$ Nul A. $A(\lambda\mathbf{u}) = \lambda A\mathbf{u} = \lambda\mathbf{0} = \mathbf{0}$.
Example 1. Find an **explicit description** of Nul A where $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$.

Sol Row reduction to solve $A\mathbf{x} = 0$; $\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \sim^{(1)/3} \begin{bmatrix} 1 & 2 & 2 & 1 & 3 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix}$
$\sim_{(2)-6(1)} \begin{bmatrix} 1 & 2 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix} \sim^{(1)-2(2)} \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$

Hence $A\mathbf{x}{=}\mathbf{0} \Leftrightarrow \begin{cases} x_1 + 2x_2 + 13x_4 + 33x_5 = 0 \\ x_3 - 6x_4 - 15x_5 = 0 \end{cases}$. x_2, x_4, x_5 are free so the sol. is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

Hence Nul $A =$ Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, is the span of the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ above.
Question: $\mathbf{u}, \mathbf{v}, \mathbf{w}$ obtained in this way are automatically linearly independent. Why?

$$c_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + c_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_2 =?,\ c_4 =?,\ c_5 =?$$

Definition. A **basis** for a vector space V is a set of vectors $\mathbf{v}_1, \dots \mathbf{v}_n$ in V which is both linearly independent and spans V.
We just obtained a basis for the null space of this matrix!

The **column space** of an $m{\times}n$ matrix $A = [\,\mathbf{a}_1 \ \cdots \ \mathbf{a}_n\,]$ is the set of all linear combinations of its column vectors; Col $A{=}$Span$\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}{=}\{\mathbf{y};\ \mathbf{y}{=}A\mathbf{x},\ \text{for some } \mathbf{x}\}$.

Theorem. The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.
Proof. In the previous section we showed that the span of any set of vectors is a subspace.

Example 2. Describe the column space of $A = [\,\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3 \ \ \mathbf{a}_4\,] = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}$

Solution. Col $A =$ Span$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$.
Example 2 continued. Do the columns of A form a basis of the column space, that is are the columns of A linearly independent?

Solution. The columns are linearly dependent if $A\mathbf{x} = 0$ only has a nontrivial solution. Row reduction on the augmented matrix gives that x_2, x_4 are free variables;

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 2 & 4 & -1 & 3 & 0 \\ 3 & 6 & 2 & 22 & 0 \\ 4 & 8 & 0 & 16 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 2 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 \quad + \quad 4x_4 = 0 \\ \qquad\qquad x_3 + 5x_4 = 0 \end{cases}$$

$\Rightarrow$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ -5x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

Notice that we have found a basis for the null space, and we see that $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, so the columns are linearly dependent.
Indeed, setting $x_2 = 1$, $x_4 = 0$ we get $-2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$.
Setting $x_2 = 0$, $x_4 = 1$, we get $-4\mathbf{a}_1 - 5\mathbf{a}_3 + \mathbf{a}_4 = \mathbf{0}$

Quest Can we find a linearly independent subset of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ that span Col A?

Idea: We use a basis for the null space to find a basis for the column space!
Using the **linear dependency relations** $-2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $-4\mathbf{a}_1 - 5\mathbf{a}_3 + \mathbf{a}_4 = \mathbf{0}$ we see that $\mathbf{a}_2 = 2\mathbf{a}_1$ and $\mathbf{a}_4 = 4\mathbf{a}_1 + 5\mathbf{a}_3$.
If $\mathbf{y} \in \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ then

$$\mathbf{y} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 + c_4\mathbf{a}_4 = ?\mathbf{a}_1 + ?\mathbf{a}_3.$$

Do $\{\mathbf{a}_1, \mathbf{a}_3\}$ span? Are they linearly independent?

Definition: If V and W are vector spaces, then a map $T : V \to W$ which assigns to each $\mathbf{v} \in V$ a unique vector $T\mathbf{v} \in W$m is a **linear transformation** if

(i) $\quad T(\mathbf{u} + \mathbf{v}) = T\mathbf{u} + T\mathbf{v} \quad$ for all $\mathbf{u}, \mathbf{v} \in V$

(ii.) $\quad T(c\mathbf{u}) = cT(\mathbf{u}) \quad$ for all $c \in \mathbb{R}$, $\mathbf{v} \in V$

The **kernel** or **null space** of T is the set of all $\mathbf{u} \in V$ such that $T\mathbf{u} = \mathbf{0}$. It is a linear subspace of V.
The **range** of T is the set of all vectors of the form $\mathbf{v}$ where $\mathbf{v} \in V$. It is a linear subspace of W.
Under the mapping T,

$$\begin{array}{rcl} \text{Kernel}\, T & \to & \mathbf{0} \\ \text{Complement to the kernel of } T & \to & \text{Range}\, T \end{array}$$

Quiz. If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ and T is multiplication by the $m \times n$ matrix A, then identify the kernel and range of T in terms of A.

Example. Identify the kernel and range of the linear map given by multiplication by the matrix in the previous example.