Lecture 11: 4.3 Linearly independent sets: Basis.

2.3 Characterizations of Invertible Matrices.

Invertible Matrix Theorem (IMT)
Let A be a given $n \times n$ matrix. Then the following are equivalent:
a) A is invertible.
b) A is row equivalent to I.
c) A has n pivot positions
d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e) The columns of A are linearly independent.
f) The linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one.
g) The equations $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$.
h) The columns of A span $\mathbf{R}^n$.
i) The linear transformation $\mathbf{x} \to A\mathbf{x}$ is onto.
j) There is an $n \times n$ matrix C such that $CA = I$.
k) There is an $n \times n$ matrix D such that $AD = I$.
l) A^T is invertible.

Basis: Let H be a subspace of a vector space V. $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ is called a **basis** for H if
(i) $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ are linearly independent, and (ii) $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ span H.

Ex 3 Show that $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $\mathbf{R}^3$, the **standard basis**
Sol They span $\mathbf{R}^3$ since any vector can be written $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$.
They are linearly independent since $x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \mathbf{x} = \mathbf{0}$ implies $x_1 = x_2 = x_3 = 0$.
Ex 4 Show that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ is a basis for $\mathbf{R}^3$.
Sol By the Invertible Matrix Theorem (IMT) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent and span $\mathbf{R}^3$ if and only if the matrix $A = [\,\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3\,]$ is invertible.
This by IMT is the same as that A has 3 pivots, which is seen to be the case;

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

Question: Why lin. indep. if 3 pivots? ($A\mathbf{x} = \mathbf{0}$) Why span if 3 pivots? ($A\mathbf{x} = \mathbf{b}$)
Ex 5 Explain why not basis: (a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} \right\}$, (b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$.
Sol (a) Linearly dependent since more vectors than rows. (b) Since the matrix with the two vectors as columns does not have a pivot in every row they do not span $\mathbf{R}^3$.
Ex 1 The three vectors in Ex 1 from Lecture 10 about section 4.2, form a basis for Nul A, since any solution to $A\mathbf{x} = \mathbf{0}$ can be written as a linear combination of them and since they are linearly independent.

The Spanning Theorem A basis can be constructed from a spanning set by discarding vectors which are linear combinations of the preceding vectors in the set.

Ex 2 Find a basis for Col A, where

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

is as in Ex 2 of Lecture 10 about section 4.2

$$[\mathbf{a}_1 \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4] = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = [\mathbf{b}_1 \mathbf{b}_2\ \mathbf{b}_3\ \mathbf{b}_4]$$

$\mathbf{b}_2 = 2\mathbf{b}_1$ and $\mathbf{a}_2 = 2\mathbf{a}_1$ $\mathbf{b}_4 = 4\mathbf{b}_1 + 5\mathbf{b}_3, \mathbf{a}_4 = 4\mathbf{a}_1 + 5\mathbf{a}_3$ $\mathbf{b}_1$ and $\mathbf{b}_3$ are not multiples. $\mathbf{a}_1$ and $\mathbf{a}_3$ are not multiples.
Elementary row operations do not affect linear dependency relations among the columns, since $A\mathbf{x} = \mathbf{0} \Leftrightarrow B\mathbf{x} = \mathbf{0}$. Hence $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$.

Th The pivot columns of A form a basis for Col A.

To form a basis for Nul A, use row operations to find the reduced row echelon form $[\,A\ \mathbf{0}\,] \sim [\,B\ \mathbf{0}\,]$ and use it to find the general solution of $A\mathbf{x} = \mathbf{0}$ in parametric form. The vectors found in parametric form is a basis.
To find a basis for Col A, use row operations to find the reduced row echelon form $A \sim B$ and use it to find the pivot columns. The pivot columns of A form a basis.