Lecture 14: 4.7 Change of basis. Recall that if $\mathcal{B}=\{\mathbf{b}_1,...,\mathbf{b}_n\}$ is a basis, then there is a unique way to write any $\mathbf{x}$ as $\mathbf{x}=c_1\mathbf{b}_1+\cdots+c_n\mathbf{b}_n$. The vector

$$[\mathbf{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is called the **$\mathcal{B}$-coordinate vector of x**. If $\mathbf{b}_1,...,\mathbf{b}_n$ are vectors in $\mathbf{R}^n$ then

$$\mathbf{x} = P_\mathcal{B}\,[\mathbf{x}]_\mathcal{B}\,, \qquad \text{where} \qquad P_\mathcal{B} = [\,\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n\,]$$

Ex Suppose we have two basis $\mathcal{B}=\{\mathbf{b}_1,\mathbf{b}_2\}$ and $\mathcal{C}=\{\mathbf{c}_1,\mathbf{c}_2\}$ that are related by

$$\mathbf{b}_1 = 3\mathbf{c}_1 + 5\mathbf{c}_2, \qquad \mathbf{b}_2 = \mathbf{c}_1 + 2\mathbf{c}_2$$

If $[\mathbf{x}]_\mathcal{B}=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, i.e. $\mathbf{x}=\mathbf{b}_1-2\mathbf{b}_2$, find $[\mathbf{x}]_\mathcal{C}$, i.e. (y_1,y_2) such that $\mathbf{x}=y_1\mathbf{c}_1+y_2\mathbf{c}_2$.
Sol

$$[\mathbf{x}]_\mathcal{C} = [\mathbf{b}_1-2\mathbf{b}_2]_\mathcal{C} = [\mathbf{b}_1]_\mathcal{C} - 2\,[\mathbf{b}_2]_\mathcal{C} = [\ [\mathbf{b}_1]_\mathcal{C}\ [\mathbf{b}_2]_\mathcal{C}\ ]\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

since $[\mathbf{b}_1]_\mathcal{C}=\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $[\mathbf{b}_2]_\mathcal{C}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Th Let $\mathcal{B}=\{\mathbf{b}_1,\ldots,\mathbf{b}_n\}$ and $\mathcal{C}=\{\mathbf{c}_1,\ldots,\mathbf{c}_n\}$ be two basis for a vector space V. Then there is a unique matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ such that

$$[\mathbf{x}]_\mathcal{C} = P_{\mathcal{C}\leftarrow\mathcal{B}}\,[\mathbf{x}]_\mathcal{B}$$

The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$:

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [\ [\mathbf{b}_1]_\mathcal{C} \quad \cdots \quad [\mathbf{b}_n]_\mathcal{C}\ ]$$

The matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is called the **change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$**.
Draw the diagram of $\mathbf{x}\in V$ and its coordinates in the two basis in $\mathbf{R}^n$.
The inverse change of variables must satisfy

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}^{-1},$$

since $P_{\mathcal{B}\leftarrow\mathcal{C}}P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the identity transformation.
If $\mathcal{C}$ is the standard basis $\mathcal{E}=\{\mathbf{e}_1,\cdots,\mathbf{e}_n\}$ then the change of coordinates

$$P_{\mathcal{E}\leftarrow\mathcal{B}} = P_\mathcal{B} = [\,\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n\,].$$

In general one can write

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}\leftarrow\mathcal{E}}P_{\mathcal{E}\leftarrow\mathcal{B}} = P_{\mathcal{E}\leftarrow\mathcal{C}}^{-1}P_{\mathcal{E}\leftarrow\mathcal{B}} = P_\mathcal{C}^{-1}P_\mathcal{B}$$

In other words we can express a vector in the different coordinate systems

$$z_1\mathbf{c}_1 + \cdots + z_n\mathbf{c}_n = y_1\mathbf{b}_1 + \cdots + y_n\mathbf{b}_n = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$$

and the matrix for the transformation $[\mathbf{x}]_{\mathcal{B}} = (y_1, \dots, y_n) \to [\mathbf{x}]_{\mathcal{C}} = (z_1, \dots, z_n)$ can be obtained by first going $(y_1, \dots, y_n) \to (x_1, \dots, x_n)$ and then $(x_1, \dots, x_n) \to (z_1, \dots, z_n)$. The matrix for the last transformation is easiest obtained as the inverse of the matrix for $(z_1, \dots, z_n) \to (x_1, \dots, x_n)$.

Ex Let $\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.
Find the change of coordinate matrix from the coordinates in the $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ basis to the coordinates in the $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ basis.

Sol $P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix}$

5.4 Matrix of a linear transformation.

Theorem. Let V be an n-dimensional vector space and let W be an m-dimensional vector space. Suppose $T : V \to W$ is a linear transformation from V to W, that is

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \qquad T(c\mathbf{u}) = cT(\mathbf{u}), \qquad \mathbf{u}, \mathbf{v} \in V, \quad c \in \mathbb{R}.$$

Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_m\}$ is a basis for W. Then

$$[T(\mathbf{x})]_{\mathcal{C}} = M\,[\mathbf{x}]_{\mathcal{B}}, \qquad M = [[T(\mathbf{b}_1)]_{\mathcal{B}}\ [T(\mathbf{b}_2)]_{\mathcal{B}}\ \dots\ [T(\mathbf{b}_n)]_{\mathcal{B}}]$$

Proof.

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \Rightarrow \quad \mathbf{x} = x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n$$

$$\Rightarrow \quad T(\mathbf{x}) = x_1 T(\mathbf{b}_1) + \cdots + x_n T(\mathbf{b}_n)$$

$$\Rightarrow \quad [T(\mathbf{x})]_{\mathcal{C}} = x_1[T(\mathbf{b}_1)_{\mathcal{C}} + \dots + x_n[T(\mathbf{b}_n)]_{\mathcal{C}} = M \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = M\,[\mathbf{x}]_{\mathcal{B}}.$$

Expressing a linear transformation in terms of different bases.

Ex Let L be the line in $\mathbf{R}^2$ that is spanned by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
Let T be the linear transformation that projects any vector orthogonally onto L. Find the matrix A for T in the standard coordinate system.

Sol We now pick a coordinate system $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ with $\mathbf{b}_1$ parallel to the line and $\mathbf{b}_2$ perpendicular to the line

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \mathbf{b}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

If $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ then $T(\mathbf{x}) = c_1\mathbf{b}_1$. Equivalently, if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ then $[T(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$:

$$[T(\mathbf{x})]_{\mathcal{B}} = B\,[\mathbf{x}]_{\mathcal{B}}\,, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The matrix B for T in the $\mathcal{B}$ coordinate system is hence very simple.
The matrix for A for T in the standard coordinates is more complicated but one can calculate it from B:

$$\begin{array}{ccc} \mathbf{x} & \xrightarrow{A} & T(\mathbf{x}) \\ {\scriptstyle P_{\mathcal{B}}}\Big\uparrow & & \Big\uparrow{\scriptstyle P_{\mathcal{B}}} \\ [\mathbf{x}]_{\mathcal{B}} & \xrightarrow{B} & [T(\mathbf{x})]_{\mathcal{B}} \end{array}\,,$$

where $P_{\mathcal{B}} = [\,\mathbf{b}_1 \ \ \mathbf{b}_2\,] = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ and $P_{\mathcal{B}}^{-1} = \dfrac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$. Hence

$$A = P_{\mathcal{B}} B P_{\mathcal{B}}^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{10} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$