Lecture 15: 3.1 Determinants. Recall the determinant of a 2 × 2 matrix:

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$.

We showed that A is invertible if and only if $\det A \neq 0$.

Most people in the class probably already learned how to compute the determinant of a 3 × 3 matrix:

Example. Find the determinant of $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

Solution.

$$\det(A) = 1\begin{vmatrix} 4 & -1 \\ -2 & 1 \end{vmatrix} - 5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & -4 \\ 0 & -2 \end{vmatrix} = -2 - 0 + -4 = -6.$$

You may have learned that you can compute the determinant by expanding around the first column rather than the first row:

Example. Compute the previous determinant this way.

Solution.

$$\det(A) = 1\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 2\begin{vmatrix} 5 & 1 \\ -2 & 0 \end{vmatrix} + 0\begin{vmatrix} 5 & 1 \\ 4 & -1 \end{vmatrix} = -2 - 4 + 0 = -6.$$

Definition. Let $A = (a_{ij})$ be an $n \times n$ matrix and let A_{ij} denote the $(n-1)\times(n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. The **cofactors** are

$$C_{ij} = (-1)^{i+j}\det(A_{ij})$$

and the **determinant** is

$$\det A = a_{11}C_{11} + ... + a_{n1}C_{n1}. \tag{2}$$

The definition is recursive; it assumes that we already defined the $(n-1)\times(n-1)$ determinants, but we can use the definition repeatedly to reduce to 2×2 determinants that we defined by (1), or we can reduce to 1×1 determinants $\det(a) = a$.

The signs above are determined by

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdot \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example. Find the determinant below. **Solution**. Expanding along the first column;

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} - 0 + 0 - 0 = 1 \cdot 2\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 2(2 \cdot 5 - 1 \cdot 3) = 14.$$

Theorem. If A is triangular the determinant is equal to the product of the diagonal elements.

This is easy to see. We illustrate the proof with an example:

Example. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{vmatrix} = 1 \begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 1 \cdot 2 \begin{vmatrix} 2 & 1 \\ 0 & 5 \end{vmatrix} = 1 \cdot 2 \cdot 2 \cdot 5 = 20.$

3.2 Properties of determinants.

Theorem (Effect of row operations on determinants):.
I) Interchanging two rows changes the sign.
II) Multiply a row by a nonzero constant α multiplies the determinant by α.
III) Add a multiple of one row to another does not change the determinant.
Proof: We deal with the 3×3 case but the general case is the same.
I) Interchanging two rows changes the sign:
2×2: Let E be the elementary matrix that interchanges two rows

$$EA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$$

Then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

$$\det(EA) = \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{11}a_{22} = -\det(A)$$

3×3: Let E be the elementary matrix that interchanges row 2 and row 3:

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Then

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$\det(EA) = a_{11} \begin{vmatrix} a_{32} & a_{33} \\ a_{22} & a_{23} \end{vmatrix} - a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -\det(A),$$

since the 2×2 determinants changes sign when we interchange the rows.
II) Multiply a row by a nonzero constant α multiplies the determinant by α.

$$EA = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We have

$$\det(EA) = \alpha a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} \alpha a_{12} & \alpha a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} \alpha a_{12} & \alpha a_{13} \\ a_{22} & a_{23} \end{vmatrix} = \alpha \det(A)$$

III) Add a multiple of one row to another does not change the determinant. To see this, we first remark that the determinant is linear in the first row. Indeed,

$$\begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

since both sides equal $(a_{11}+b_{11})a_{22} + (a_{12}+b_{12})a_{21}$, and so

$$\begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

since both sides equal

$$(a_{11}+b_{11}) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12}+b_{12} & a_{13}+b_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12}+b_{12} & a_{13}+b_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

Now we notice that if A has two repeated rows then $\det A = 0$. Indeed, using the rules already proved we have

$$\begin{aligned} \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= - \begin{vmatrix} -a_{31} & -a_{32} & -a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{vmatrix} = - \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \end{aligned}$$

Hence

$$\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.$$

Finally writing

$$EA = \begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}+\beta a_{31} & a_{12}+\beta a_{32} & a_{13}+\beta a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

we deduce that

$$\begin{aligned} \det(EA) = \begin{vmatrix} a_{11}+\beta a_{31} & a_{12}+\beta a_{32} & a_{13}+\beta a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \beta \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \det A. \end{aligned}$$

Corollary. If $A \sim B$ under row reduction then $\det A \neq 0 \Leftrightarrow \det B \neq 0$.

Proof. $B = E_k \cdots E_1 A$, where E_i are elementary matrices. Just apply the previous Theorem k times.

Important Corollary. If A is an $n \times n$ matrix then $\det A \neq 0$ if and only if A is invertible.

Proof. The matrix A can be reduced to row echelon form U. But if U is invertible then it is the identity matrix and $\det U = 1$, while if U is singular then it is triangular with at least one zero on the diagonal and $\det U = 0$. Thus

$$A \text{ invertible} \;\Leftrightarrow\; U \text{ invertible} \;\Leftrightarrow\; \det U \neq 0 \;\Leftrightarrow\; \det A \neq 0.$$

We will see later that in fact we can expand along any row or column:
Theorem. For any i and j we have

$$\det A = a_{i1}C_{i1} + ... + a_{in}C_{in} = a_{1j}C_{1j} + ... + a_{nj}C_{nj}$$