Lecture 18 5.1-5.2 Eigenvectors and Eigenvalues.

Let A be an $n \times n$ matrix. A scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq 0$ is called an **eigenvalue** and a corresponding vector $\mathbf{x}$ is called an **eigenvector**.
λ is an eigenvalue if and only if there is $\mathbf{x} \neq \mathbf{0}$ such that

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

The set of all solutions is called the **eigenspace** corresponding to eigenvalue λ. Existence of a nontrivial solution $\mathbf{x}$ is equivalent to that A is not invertible which is equivalent to

$$p(\lambda) \equiv \det(A - \lambda I) = 0.$$

This is called the **characteristic polynomial** for the matrix A, since indeed it is a polynomial, and its roots are exactly the eigenvalues of A.

Ex 1 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$.

Sol The eigenvalues are solution of the characteristic equation:
$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 2^2 = (1-\lambda-2)(1-\lambda+2) = 0$
The eigenvalues are $\lambda_1 = -1, \lambda_2 = 3$. The eigenvectors are solutions to $(A - \lambda_i)\mathbf{x}_i = \mathbf{0}$:

$$(A - \lambda_1 I)\mathbf{x}_1 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{r} 2x_1 - 2x_2 = 0 \\ -2x_1 + 2x_2 = 0 \end{array} \quad \Leftrightarrow \quad \mathbf{x}_1 = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I)\mathbf{x}_2 = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{r} -2x_1 - 2x_2 = 0 \\ -2x_1 - 2x_2 = 0 \end{array} \quad \Leftrightarrow \quad \mathbf{x}_2 = \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Ex 2 Find the eigevales and eigenspaces of $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Sol Characteristic polynomial $(2-\lambda)^2(4-\lambda)$.
$A - 2I = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ and $(A - 2I)\mathbf{x} = \mathbf{0}$ has augmented matrix $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Leftrightarrow \quad x_1 - x_2 - x_3 = 0$
and hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ spann the eigenspace.
$A - 4I = \begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$, augmented matrix $\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$
$\Leftrightarrow \begin{cases} x_1 = 0, \\ x_2 - x_3 = 0 \end{cases}$ and hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Ex 3 Let T be the linear transformation rotating a vector an angle θ. The matrix for T is $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Find the eigenvectors and eigenvalues of T.
Sol Unless θ is a multiple of π it does not have any real eigenvalues and eigenvectors. If θ is a multiply of π the eigenvalues are ± 1.

Ex 3 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Sol This is the matrix for a rotation with scaling: $A = \sqrt{2}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\theta = \pi/4$ and can not have any real eigenvectors unless the rotation a multiple of π.
The eigenvalues are solution of:
$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1^2 = (1-\lambda-i)(1-\lambda+i) = 0,$
i.e. $\lambda = \lambda_1 = 1 + i$, or $\lambda = \lambda_2 = 1 - i$. The eigenvectors are solutions to:

$$(A - \lambda_1 I)\mathbf{x}_1 = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} ix_1 + x_2 = 0 \\ -x_1 + ix_2 = 0 \end{matrix} \quad \Leftrightarrow \quad \mathbf{x}_1 = \alpha\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I)\mathbf{x}_1 = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} -ix_1 + x_2 = 0 \\ -x_1 - ix_2 = 0 \end{matrix} \quad \Leftrightarrow \quad \mathbf{x}_2 = \beta\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Even though in many applications we are looking for real solutions the complex solutions can still be helpful on the way towards a final answer as we shall see.

Th If B is similar to A, i.e. $B = S^{-1}AS$ then A and B have the same characteristic polynomial and hence the same eigenvalues.
Pf Since we can write $I = S^{-1}S = S^{-1}IS$ we get

$$\begin{aligned} \det(B - \lambda I) &= \det\left(S^{-1}AS - \lambda S^{-1}IS\right) = \det\left(S^{-1}(A - \lambda I)S\right) \\ &= \det S^{-1}\,\det(A - \lambda I)\,\det S = \det(A - \lambda I) \end{aligned}$$

By the product rules for determinants: $(\det(CD) = \det C\,\det D)$.

This theorem says something very important; that the eigenvalues does not depend on in which coordinate system we view a linear transformation, and hence describe some fundamental property of the linear transformation.