Lecture 20: 5.4-5.6.
From last time:

Ex If possible, diagonalize $A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

Sol The eigenvalues $\det(A - \lambda I) = (\lambda - 2)^2(\lambda - 4) = 0$.

Basis for $\lambda = 2$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Basis for $\lambda = 4$: $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$.

There are not three linearly independent eigenvectors so A can not be diagonalized.

Ex If possible, diagonalize $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

Sol The eigenvalues $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)^2(1-\lambda) = 0$.

Basis for $\lambda = 1$: $\mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Basis for $\lambda = 2$: $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Construct $P = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3\,] = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. $A = PDP^{-1}$.

5.4 Expressing a linear transformation in terms of different bases.

Ex 2 Let L be the line in $\mathbf{R}^2$ that is spanned by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
Let T be the linear transformation that projects any vector orthogonally onto L. Find the matrix A for T in the standard coordinate system.

Sol Since the projection leaves the line invariant the vector $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ must be an eigenvector with eigenvalue 1: $A\mathbf{x}_1 = \mathbf{x}_1$. Moreover, since the orthogonal vector $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is mapped to $\mathbf{0}$ its also an eigenvector with eigenvalue 0: $A\mathbf{x}_2 = \mathbf{0} = 0 \cdot \mathbf{x}_2$.
Hence $A[\mathbf{x}_1\ \mathbf{x}_2] = [\mathbf{x}_1\ \mathbf{0}] = [\mathbf{x}_1\ \mathbf{x}_2] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $A = PDP^{-1}$
where $P = [\,\mathbf{x}_1\ \mathbf{x}_2\,] = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
Hence

$$A = PDP^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{10} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

If we express If $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$, in terms of the basis $\mathcal{B}$ of eigenvectors then $A\mathbf{x} = c_1\mathbf{x}_1$. The matrix for T in these coordinates is D. The change of basis matrix is P.

$$\begin{array}{ccc} c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{x} & \xrightarrow{A} & A\mathbf{x} = c_1\mathbf{x}_1 \\ {\scriptstyle P}\uparrow & & \uparrow{\scriptstyle P} \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}} & \xrightarrow{D} & [A\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \end{array},$$

5.5 Complex Eigenvalues.

Ex 3 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Sol The eigenvalues are and eigenvectors are complex:
$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1^2 = (1-\lambda-i)(1-\lambda+i) = 0$, The complex eigenvector for $\lambda_1 = 1+i$ is $\mathbf{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ and for $\lambda_2 = 1-i$ is $\mathbf{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$. Hence we get

$$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = PDP^{-1}, \qquad P = \begin{vmatrix} -i & i \\ 1 & 1 \end{vmatrix} \qquad D = \begin{vmatrix} 1+i & 0 \\ 0 & 1-i \end{vmatrix}.$$

5.6 Discrete Dynamical Systems.

Ex Denote the owl and rat population at time k by $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$. Suppose

$$O_{k+1} = 0.5\, O_k + 0.4\, R_k$$
$$R_{k+1} = -p\, O_k + 1.1\, R_k$$

where $p = 0.104$, or $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where $A = \begin{bmatrix} 0.5 & 0.4 \\ -.104 & 1.1 \end{bmatrix}$. The eigenvalues for the matrix A are $\lambda_1 = 1.02$ and $\lambda_2 = 0.58$ and the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. An initial $\mathbf{x}_0$ can be written $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then for $k \geq 0$

$$\mathbf{x}_k = c_1 A^k \mathbf{v}_1 + c_2 A^k \mathbf{v}_2 = c_1 \lambda_1^k \mathbf{v}_1 + c_2 \lambda_2^k \mathbf{v}_2 = c_1 (1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2 (0.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

As k becomes large the first state will dominate and the other will go to $\mathbf{0}$ unless the initial conditions are such that $c_1 = 0$ in which case the whole solution goes to $\mathbf{0}$.