Lecture 21: 6.1 Inner product, Length and Orthogonality.

The **inner product** or **dot product** between two vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ is

$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = [\, x_1 \ \cdots \ x_n \,] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n$. The dot product satisfy $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ and $\mathbf{x} \cdot \mathbf{x} > 0$, if $\mathbf{x} \neq 0$.

The **length** of the vector $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + \cdots + x_n^2}$

We know this is the Euclidean length in two dimensions by the Phytagorean law.

The **distance** between $\mathbf{x}$ and $\mathbf{y}$ is $\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|$.

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are called **orthogonal** if $\mathbf{x} \cdot \mathbf{y} = 0$.

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are colled **perpendicular** if $\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \operatorname{dist}(\mathbf{x}, -\mathbf{y})$.

$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|^2 = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\mathbf{x} \cdot \mathbf{y}$,

$\operatorname{dist}(\mathbf{x}, -\mathbf{y}) = \|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\mathbf{x} \cdot \mathbf{y}$.

It follows that $\mathbf{x} \cdot \mathbf{y} = 0$ if $\mathbf{x}$ and $\mathbf{y}$ are perpendicular so it is the same as orthogonal.

The **Pythagorean law:** $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x} - \mathbf{y}\|^2$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

$\mathbf{z}$ is said to be **orthogonal to** a subspace W if it is orthogonal to every vector in W. The set of all vectors $\mathbf{z}$ that are orthogonal to a subspace $W \subset \mathbf{R}^n$ is called the **orthogonal complement** of W and is denoted by $W^\perp$.

$$W^\perp = \{\mathbf{z} \in \mathbf{R}^n;\ \mathbf{z} \cdot \mathbf{y} = 0, \quad \text{for every} \quad \mathbf{y} \in W\}$$

Ex If W is plane through the origin in $\mathbf{R}^3$ and L is the line through the origin perpendicular to W, then $W^\perp = L$. In fact, clearly $L \subset W^\perp$ since L is perpendicular to W and any vector not in L is not perpendicular to W. Similarly $L^\perp = W$.

Ex If $V = \{\mathbf{x} \in \mathbf{R}^3;\ \mathbf{x} = \alpha(1,1,1), \text{ for some } \alpha\}$ then $V^\perp = \{\mathbf{y} \in \mathbf{R}^3;\ \alpha(1,1,1) \cdot \mathbf{y} = 0, \text{ for every } \alpha\} = \{\mathbf{y} \in \mathbf{R}^3;\ y_1 + y_2 + y_3 = 0\}$.

(1) If $W = \operatorname{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ then $\mathbf{z} \in W^\perp$ if and only if $\mathbf{z} \cdot \mathbf{v}_1 = \cdots = \mathbf{z} \cdot \mathbf{v}_k = 0$.

(2) $W^\perp$ is a subspace.

Th Let A be an $m \times n$ matrix. Then $(\operatorname{Row} A)^\perp = \operatorname{Nul} A$ and $(\operatorname{Col} A)^\perp = \operatorname{Nul} A^T$.

Pf If $\mathbf{x} \in \operatorname{Nul} A$ then

$$\begin{matrix} \mathbf{r}_1 \to \\ \mathbf{r}_2 \to \\ \vdots \\ \mathbf{r}_m \to \end{matrix} \begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

so $\mathbf{x}$ is orthogonal to Row , A since its orthogonal to $\mathbf{r}_1, \dots, \mathbf{r}_m$.

Ex Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{bmatrix}$. Basis for $\operatorname{Nul} A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, so $\operatorname{Nul} A$ is a plane in $\mathbf{R}^3$. Basis for $\operatorname{Row} A = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ so $\operatorname{Row} A$ is a line in $\mathbf{R}^3$. Basis for $\operatorname{Col} A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $\operatorname{Col} A$ is a line in $\mathbf{R}^2$. Basis for $\operatorname{Nul} A^T = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, so $\operatorname{Nul} A^T$ is a line in $\mathbf{R}^2$.