Lecture 22: 6.2 Orthogonal Sets.

A set of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ in called and **orthogonal set** if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ when $i \neq j$.

Ex Show that $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthogonal set? **Sol:**

$\mathbf{u}_1 \cdot \mathbf{u}_2 = 1 \cdot 1 + -1 \cdot 1 + 0 \cdot 0 = 0$, $\quad \mathbf{u}_1 \cdot \mathbf{u}_3 = 1 \cdot 0 + -1 \cdot 0 + 0 \cdot 1 = 0$, $\quad \mathbf{u}_2 \cdot \mathbf{u}_3 = 1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0$

Th Suppose that $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbf{R}^n$ and $W = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$. Then S is a linearly independent set and a basis for W.
Pf Suppose

$$c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_p \mathbf{u}_p = \mathbf{0},$$

$$\big(c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_p \mathbf{u}_p\big) \cdot \mathbf{u}_1 = 0,$$

$$c_1 \mathbf{u}_1 \cdot \mathbf{u}_1 + c_2 \mathbf{u}_2 \cdot \mathbf{u}_1 + \cdots + c_p \mathbf{u}_p \cdot \mathbf{u}_1 = 0,$$

$$c_1 \mathbf{u}_1 \cdot \mathbf{u}_1 = 0,$$

$c_1 = 0$ since $\mathbf{u}_1 \cdot \mathbf{u}_1 > 0$. Similarly $c_2 = \cdots = c_p = 0$, so S is a linearly independent set.

An **orthogonal basis** $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ for a subspace W is a basis that is also orthogonal, i.e. $\mathbf{u}_i \cdot \mathbf{u}_j = 0$, whenever $i \neq j$.

Th If $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal basis for a subspace W and $\mathbf{y} \in W$, then

$$\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_p \mathbf{u}_p, \qquad \text{where} \quad c_i = \frac{\mathbf{y} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i}$$

Pf

$\mathbf{y} \cdot \mathbf{u}_1 = \big(c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_p \mathbf{u}_p\big) \cdot \mathbf{u}_1 = c_1 \mathbf{u}_1 \cdot \mathbf{u}_1 + c_2 \mathbf{u}_2 \cdot \mathbf{u}_1 + \cdots + c_p \mathbf{u}_p \cdot \mathbf{u}_1 = c_1 \mathbf{u}_1 \cdot \mathbf{u}_1$

Hence $c_1 = \dfrac{\mathbf{u}_1 \cdot \mathbf{y}}{\mathbf{u}_1 \cdot \mathbf{u}_1}$ and similarly $c_2 = \dfrac{\mathbf{u}_2 \cdot \mathbf{y}}{\mathbf{u}_2 \cdot \mathbf{u}_2}$, $\quad \cdots \quad c_p = \dfrac{\mathbf{u}_p \cdot \mathbf{y}}{\mathbf{u}_p \cdot \mathbf{u}_p}$

Ex Write $\mathbf{y} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Sol Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis it follows from the previous theorem that $\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3$, where

$$c_1 = \frac{\mathbf{u}_1 \cdot \mathbf{y}}{\mathbf{u}_1 \cdot \mathbf{u}_1} = -2, \qquad c_2 = \frac{\mathbf{u}_2 \cdot \mathbf{y}}{\mathbf{u}_2 \cdot \mathbf{u}_2} = 5, \qquad c_3 = \frac{\mathbf{u}_3 \cdot \mathbf{y}}{\mathbf{u}_3 \cdot \mathbf{u}_3} = 4$$

Hence $\mathbf{y} = -2\mathbf{u}_1 + 5\mathbf{u}_2 + 4\mathbf{u}_3$.

We will now calculate the **orthogonal projection** of $\mathbf{y}$ onto $\mathbf{u}$.
It is a vector $\widehat{\mathbf{y}} = \alpha \mathbf{u}$ in the direction of $\mathbf{u}$, such that $\mathbf{y} - \widehat{\mathbf{y}}$ is orthogonal to $\mathbf{u}$:

$$(\mathbf{y} - \alpha \mathbf{u}) \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{y} \cdot \mathbf{u} - \alpha\, \mathbf{u} \cdot \mathbf{u} = 0 \quad \Leftrightarrow \quad \alpha = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$$

The **orthogonal projection of y onto u** is the vector $\widehat{\mathbf{y}} = \dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$. We can write

$$\mathbf{y} = \widehat{\mathbf{y}} + \mathbf{z}$$

where $\mathbf{z}$ is called the **component orthogonal to u**.

Ex Let $\mathbf{y} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find the orthogonal projection $\widehat{\mathbf{y}}$ of $\mathbf{y}$ onto $\mathbf{u}$.
Write $\mathbf{y}$ as a vector parallel to $\mathbf{u}$ and a vector perpendicular to $\mathbf{u}$.
Compute the distance from $\mathbf{y}$ to the line through $\mathbf{0}$ and $\mathbf{u}$.

Sol $\widehat{\mathbf{y}} = \dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}} = \dfrac{-24+4}{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$. $\mathbf{y} = \widehat{\mathbf{y}} + (\mathbf{y} - \widehat{\mathbf{y}})$ We have $\widehat{\mathbf{y}} - \mathbf{y} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$ so $\|\widehat{\mathbf{y}} - \mathbf{y}\| = \sqrt{2^2 + 6^2} = \sqrt{40}$.

A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is called **orthonormal** if it is a orthogonal set of unit vectors i.e.

$$\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

If $W = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ then $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an **orthonormal basis** for W.
Recall that $\mathbf{v}$ is a unit vector if $\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = 1$.

Suppose that $U = [\,\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\,]$, where $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set. Then

$$U^T U = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} [\,\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\,] = \begin{bmatrix} \mathbf{u}_1^T\cdot\mathbf{u}_1 & \mathbf{u}_1^T\cdot\mathbf{u}_2 & \mathbf{u}_1^T\cdot\mathbf{u}3 \\ \mathbf{u}_2^T\cdot\mathbf{u}_1 & \mathbf{u}_2^T\cdot\mathbf{u}_2 & \mathbf{u}_2^T\cdot\mathbf{u}_3 \\ \mathbf{u}_3^T\cdot\mathbf{u}_1 & \mathbf{u}_3^T\cdot\mathbf{u}_2 & \mathbf{u}_3^T\cdot\mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A matrix such that $U^T U = I$ is called **orthogonal** (not orthonormal)
U is orthogonal if and only if the columns are orthonormal.
We have that multiplying by an orthogonal matrix preserves lengths and angles between vectors since $(U\mathbf{x})\cdot(U\mathbf{y}) = (U\mathbf{x})^T U\mathbf{y} = \mathbf{x}^T U^T U\mathbf{y} = \mathbf{x}^T I\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x}\cdot\mathbf{y}$