MATH 20F LINEAR ALGEBRA

Lecture 25: 6.7-6.8: Inner products and Fourier series.

Def An **inner product** on a vector space is a function that for each pair of vectors gives a real number: $V \ni \mathbf{x}, \mathbf{y} \to \langle \mathbf{x}, \mathbf{y} \rangle \in \mathbf{R}$, satisfying:
(i) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ with equality if and only if $\mathbf{x} = 0$.
(ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.
(iii) $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z} \rangle = \alpha\langle \mathbf{x}, \mathbf{z} \rangle + \beta\langle \mathbf{y}, \mathbf{z} \rangle$.

Ex 1 $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y} = x_1y_1 + \cdots + x_ny_n$.

Ex 2 $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + 2x_2y_2 + 4x_3y_3 + \ldots \cdots + 2^n x_ny_n$.

Ex 3 $f, g \in C[a, b]$, the continuous functions on the interval $[a, b]$, and

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx$$

Def $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Def The norm is $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

Note that Phytagorean law holds: $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
Indeed, $\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle$.

Ex 4 $V = C[0, 2\pi]$ is a vector space. Let W be the subspace spanned by all trigonometric polynomials up to order n: $1, \cos t, \ldots, \cos nt, \sin t, \ldots, \sin nt$, i.e. W consists of all functions of the form

$$\frac{a_0}{2} + a_1 \cos t + \cdots + a_n \cos nt + b_1 \sin t + \cdots + b_n \sin nt$$

The basis vectors $1, \cos t, \ldots, \cos nt, \sin t, \ldots, \sin nt$, are orthogonal to each other, i.e.

$$\int_0^{2\pi} \cos mt \sin nt\, dt = 0$$

$$\int_0^{2\pi} \cos mt \cos nt\, dt = 0, \qquad \text{if } m \neq n, \qquad = \pi, \text{if } m = n,$$

$$\int_0^{2\pi} \sin mt \sin nt\, dt = 0, \qquad \text{if } m \neq n \qquad = \pi, \text{if } m = n,$$

Using Euler's formulas, $\cos mt = \dfrac{e^{imt} + e^{-imt}}{2}$, $\sin mt = \dfrac{e^{imt} - e^{-imt}}{2i}$, the proof reduces to

$$\int_0^{2\pi} e^{imt}\, dt = 0, \qquad \text{if } m \neq 0, \qquad = \pi, \text{if } m = 0,$$

Question Given $f \in C[0, 2\pi]$ which is the function $p \in W$ closest to f, i.e. such that $\|f - p\|$ is as small as possible?

Answer The orthogonal projection of f onto W.

The orthogonal decomposition theorem Let W be a subspace of a vector space V and suppose that $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal basis for W. Any $\mathbf{y} \in V$ can be written uniquely as

$$\mathbf{y} = \widehat{\mathbf{y}} + \mathbf{z},$$

where

$$\widehat{\mathbf{y}} = \frac{\langle \mathbf{y}, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 + \cdots + \frac{\langle \mathbf{y}, \mathbf{u}_p \rangle}{\langle \mathbf{u}_p, \mathbf{u}_p \rangle} \mathbf{u}_p$$

and $\mathbf{z} = \mathbf{y} - \widehat{\mathbf{y}} \in W^{\perp}$, the orthogonal complement $W^{\perp} = \{\mathbf{z} \in \mathbf{R}^n;\ \langle \mathbf{z}, \mathbf{u}_1 \rangle = 0, \dots, \langle \mathbf{z}, \mathbf{u}_p \rangle = 0\}$. $\widehat{\mathbf{y}} = \text{proj}_W \mathbf{y}$ is called the **orthogonal projection of $\mathbf{y}$ onto W**.

The best approximation theorem Let W be a subspace of a vector space V, $\mathbf{y}$ a vector and $\widehat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\widehat{\mathbf{y}}$ is the point in W closest to $\mathbf{y}$:

$$\|\mathbf{y} - \widehat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\|, \qquad \mathbf{v} \in W, \quad \mathbf{v} \neq \widehat{\mathbf{y}}.$$

Pf We can write

$$\mathbf{y} - \mathbf{v} = \mathbf{y} - \widehat{\mathbf{y}} + \widehat{\mathbf{y}} - \mathbf{v}$$

where $\mathbf{y} - \widehat{\mathbf{y}} \in W^{\perp}$ and $\widehat{\mathbf{y}} - \mathbf{v} \in W$ are orthogonal and hence by the Pythagorean theorem:

$$||\mathbf{y} - \mathbf{v}\|^2 = \|\mathbf{y} - \widehat{\mathbf{y}}\|^2 + \|\widehat{\mathbf{y}} - \mathbf{v}\|^2 > \|\mathbf{y} - \widehat{\mathbf{y}}\|^2.$$

The orthogonal projection of f onto W is given by

$$\begin{aligned}
\widehat{\mathbf{f}} &= \frac{\langle \mathbf{f}, 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle \mathbf{f}, \cos t \rangle}{\langle \cos t, \cos t \rangle} \cos t + \cdots + \frac{\langle \mathbf{f}, \cos nt \rangle}{\langle \cos nt, \cos nt \rangle} \cos nt \\
&\quad + \frac{\langle \mathbf{f}, \sin t \rangle}{\langle \sin t, \sin t \rangle} \sin t + \cdots + \frac{\langle \mathbf{f}, \sin nt \rangle}{\langle \sin nt, \sin nt \rangle} \sin nt \\
&= \left(\frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx \right) 1 + \left(\frac{1}{\pi} \int_0^{2\pi} f(x) \cos x\, dx \right) \cos t + \cdots + \left(\frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx \right) \cos nt \\
&\quad + \left(\frac{1}{\pi} \int_0^{2\pi} f(x) \sin x\, dx \right) \sin t + \cdots + \left(\frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx \right) \sin nt \\
&= a_0 + a_1 \cos t + \dots + a_n \cos nt + b_1 \sin t + \dots + b_n \sin t.
\end{aligned}$$

This is called the **nth-order Fourier approximation to the function f on the interval** $[0, 2\pi]$.

Example. Calculate the nth-order Fourier approximation to the function f defined by

$$f(x) = \begin{cases} 1 & 0 \le x \le \pi, \\ 0\pi < x \le 2\pi. & \end{cases}$$

Solution.

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{2\pi} \int_0^{\pi} 1\, dx = \frac{\pi}{2\pi} = \frac{1}{2}.$$

$$a_k \;=\; \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx \;=\; \frac{1}{\pi}\int_0^{\pi} \cos kx\,dx \;=\; \frac{1}{\pi}\left[\frac{\sin kx}{k}\right]_0^{\pi} \;=\; 0.$$

$$b_k \;=\; \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx \;=\; \frac{1}{\pi}\int_0^{\pi} \cos kx\,dx \;=\; \frac{1}{\pi}\left[\frac{-\cos kx}{k}\right]_0^{\pi} \;=\; \frac{1-\cos k\pi}{k} \;=\; \begin{cases} 0 & k \text{ even,} \\ \frac{2}{k} & k \text{ odd.} \end{cases}$$

The nth-order Fourier approximation to the function f is

$$\frac{1}{2} \;+\; \sin x \;+\; \frac{2\sin 3x}{3} \;+\; \frac{2\sin 5x}{5} \;+\; \ldots \;+\; \begin{cases} \frac{2\sin kx}{k} & k \text{ odd,} \\ \frac{2\sin(k-1)x}{k-1} & k \text{ even.} \end{cases}$$