Lecture 26: 7.1 Diagonalization of symmetric matrices.

Spectral Th. If A is a real and symmetric $n\times n$ matrix then it has an orthonormal set of n eigenvectors and hence can be diagonalized by an orthogonal matrix U. $U^{-1}AU = D$, or $A = UDU^{-1}$, where D is diagonal and $U^{-1} = U^T$.

Ex Diagonalize $A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$ with an orthogonal transformation.

Sol A is symmetric so it can be diagonalized by an orthogonal transformation. The characteristic polynomial

$$\det(A - \lambda I) = (1+\lambda)^2(5-\lambda)$$

has roots $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$. Computing eigenvectors we see that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ form a basis for the eigenspace corresponding to $\lambda = -1$. We can apply the Gram-Schmidt process to obtain an orthonormal basis. Let

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{p}_2 = \mathbf{v}_2 \cdot \mathbf{u}_1\, \mathbf{u}_1 = -\sqrt{2}\mathbf{u}_1 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_2 - \mathbf{p}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{\mathbf{v}_2 - \mathbf{p}_2}{\|\mathbf{v}_2 - \mathbf{p}_2\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The eigenspace corresponding to $\lambda_3 = 5$ is spanned by $\mathbf{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ and we set

$$\mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Hence $A = UDU^T$ where

$$U = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3\,] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}, \qquad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Th Eigenvectors belonging to distinct eigenvalues of a symmetric matrix are orthogonal. The eigenvalues of a symmetric matrix are real.
Pf Suppose $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors with eigenvalues λ_1, λ_2. Then

$$\lambda_1 \mathbf{u}_1 \cdot \mathbf{b}_2 = (A\mathbf{u}_1) \cdot \mathbf{u}_2 = (A\mathbf{u}_1)^T \mathbf{u}_2 = \mathbf{u}_1^T A^T \mathbf{u}_2 = \mathbf{u}_1^T A \mathbf{u}_2 = \lambda_2 \mathbf{u}_1^T \mathbf{u}_2 = \lambda_2 \mathbf{u}_1 \cdot \mathbf{u}_2.$$

Hence if $\lambda_1 \neq \lambda_2$ then $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$.

Now suppose that $\mathbf{u}$ is a complex eigenvector with complex eigenvalue λ. Then

$$A\mathbf{u} = \lambda \mathbf{u}.$$

Taking the complex conjugate, since A is real we get

$$A\bar{\mathbf{u}} = \bar{\lambda} \bar{\mathbf{u}}.$$

Hence $\bar{\mathbf{u}}$ is an eigenvector with eigenvalue $\bar{\lambda}$. Applying the previous argument we get

$$\lambda \bar{\mathbf{u}} \cdot \mathbf{u} = \bar{\lambda} \bar{\mathbf{u}} \cdot \mathbf{u}.$$

Since $\bar{\mathbf{u}} \cdot \mathbf{u} > 0$, we find that

$$\lambda = \bar{\lambda}$$

so λ is real.