Lecture 3: Vector Equations, Linear Combinations and Span.
In linear algebra we think of **vectors** in $\mathbf{R}^n$ as column vectors or $n \times 1$ matrices

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Addition and **scalar multiplication** are defined by

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}, \qquad \lambda\mathbf{u} = \begin{bmatrix} \lambda u_1 \\ \lambda u_2 \\ \vdots \\ \lambda u_n \end{bmatrix}, \quad \lambda \in \mathbf{R}.$$

Linear Combination: Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$, and scalars $\lambda_1, \ldots, \lambda_k$, the vector

$$\mathbf{w} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k$$

is called a **linear combination** of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$, (with weights $\lambda_1, \ldots \lambda_k$).

Question: If you give me a vector $\mathbf{w}$, and vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$, how can I figure out whether $\mathbf{w}$ is a linear combination of $\mathbf{v}_1, \ldots \mathbf{v}_k$?

Geometric Interpretation: In $\mathbf{R}^2$ and $\mathbf{R}^3$ We think of vectors as arrows with a length and a direction.

The parallelogram law says that the sum $\mathbf{u} + \mathbf{v}$ is given by placing the start of $\mathbf{v}$ where $\mathbf{u}$ ends. Check this by drawing $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 1+2 \\ 3+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

If $\lambda > 0$ then scalar multiplication $\lambda\,\mathbf{u}$ is the vector in the same direction as $\mathbf{u}$ with length λ times the length of $\mathbf{u}$. Check this by drawing $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $2\mathbf{u} = \begin{bmatrix} 2 \cdot 1 \\ 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Ex Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of $\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.
We start by drawing a net of parallelograms with sides $\mathbf{v}_1$ and $\mathbf{v}_2$. Then we see how far we should go first in the $\mathbf{v}_1$ and then in the $\mathbf{v}_2$ direction to reach first $\mathbf{a}$.

Ex Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$. Is $\mathbf{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$.
Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$.
Sol $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ if we can find scalars x_1, x_2, x_3 so

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

If we write it out we get the vector equation

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$$

If we add up the vectors to the left we get

$$\begin{bmatrix} x_1 + 4x_2 + 3x_3 \\ 2x_2 + 6x_3 \\ 3x_1 + 14x_2 + 10x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$$

i.e. we get a linear system of equations

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= -1 \\ 2x_2 + 6x_3 &= 8 \\ 3x_1 + 14x_2 + 10x_3 &= -5 \end{aligned}$$

The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 0 & 2 & 1 & -2 \end{bmatrix} (3) - 3(1) \quad \Leftrightarrow \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 1 & -2 \end{bmatrix} (2)/2 \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -5 & -6 \end{bmatrix} (3) - 2(2) \quad \Leftrightarrow \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} (3)/(-5)$$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 & -9 & -17 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} (1) - 4(2) \quad \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} (1) + 9(3) \\ (2) - 3(3) \\ \end{matrix}$$

i.e. we get the system

$$\begin{aligned} x_1 \qquad\qquad &= 1 \\ x_2 \qquad &= -2 \\ x_3 &= 2 \end{aligned}$$

and hence

$$\mathbf{b} = \mathbf{a}_1 - 2\mathbf{a}_2 + 2\mathbf{a}_3$$

Note that $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{b}$ are columns of the augmented matrix. Hence solving the vector equation $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$ is the same as solving the linear system with augmented matrix $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}]$.

In general the vector equation

$$\mathbf{b} = x_1\mathbf{a}_1 + \cdots + x_k\mathbf{a}_k$$

has the same solution set as the linear system with augmented matrix

$$[\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_k \quad \mathbf{b}]$$

i.e. $\mathbf{b}$ can be generated as a linear combination of $\mathbf{a}_1, \cdots, \mathbf{a}_k$ if and only if there is a solution to the linear system with the corresponding augmented matrix.

The **span** of the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_p$ denoted by $\mathbf{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is defined to be the set of all vectors $\mathbf{v}$ that can be written as a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_p$, i.e. all vectors that can be written

$$\mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p$$

for some scalars $x_1, \cdots, x_n$.

Ex Let $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$. Then $\mathbf{Span}\{\mathbf{v}\}$ is the line through the origin in the direction of $\mathbf{v}$.

Ex Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ Then $\mathbf{Span}\{\mathbf{u}, \mathbf{v}\}$ is the plane $x - y = 0$.

Ex Let $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ Is $\mathbf{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ a line or a plane?

Sol Since $\mathbf{v}_2 = 2\mathbf{v}_1$ we have

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = (x_1 + 2x_2)\mathbf{v}_1$$

so all linear combinations are on the line in the direction of $\mathbf{v}_1$ so $\mathbf{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a line.

Ex Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$. Is $\mathbf{b}$ in the plane spanned by the columns of A?

Sol The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 5 & 17 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 0 & -4 \end{bmatrix}$$

The system is inconsistent so $\mathbf{b}$ is not in the plane spanned by the columns of A.