

Lecture 9: 4.1 Vector Spaces.

Recall that if $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are vectors in Euclidean space we defined the **addition** $\mathbf{x} + \mathbf{y} \in \mathbf{R}^n$ and **scalar multiplication** $\lambda \mathbf{x} \in \mathbf{R}^n$. In dimensions 2 and 3 we can define these notions geometrically

The addition and scalar multiplication satisfy certain properties listed below, but these properties show up in many different contexts so rather than studying each situation individually we will study them all at once.

A set V with two operations, addition and multiplication by scalars, defined on it is called a **vector space** if the following properties hold for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ $\alpha, \beta \in \mathbf{R}$:

1. If $\mathbf{u}, \mathbf{v} \in V$ then $\mathbf{u} + \mathbf{v} \in V$. (closure under addition)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutative)
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associative)
4. There is an element $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ all $\mathbf{u} \in V$ (additive unit)
5. For each $\mathbf{u} \in V$ there is $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ (additive inverse)
6. If $\mathbf{u} \in V$ and α is a scalar then $\alpha \mathbf{u} \in V$. (closure under scalar multiplication)
7. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$ (distributive)
8. $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$ (distributive)
9. $(\alpha\beta)\mathbf{u} = \alpha(\beta \mathbf{u})$ (associative)
10. $1 \cdot \mathbf{u} = \mathbf{u}$ (multiplicative unit)

It follows from (1)-(10) that $0\mathbf{u} = \mathbf{0}$, $(-1)\mathbf{u} = -\mathbf{u}$, $c\mathbf{0} = \mathbf{0}$.

Examples.

- \mathbf{R}^n (dimension = n).
- $m \times n$ matrices; $\mathbf{R}^{m \times n}$. (dimension= mn).
- Polynomials in the variable x with real coefficients and degree at most n . A vector has the form $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ (dimension $n + 1$).
- Polynomials in the variable x with real coefficients and arbitrary degree. (dimension ∞).
- Let $I = [0, 1]$ and let $C(I, \mathbf{R})$ be the set of real valued continuous functions $I \rightarrow \mathbf{R}$ (dimension= ∞).

Question: When is a subset of a vector space a vector space itself?

We defined a vector spaces V as a set with addition and scalar multiplication that satisfy 10 axioms. However, often we have a subset of a vector space in which case we only need to check that its closed under addition and scalar multiplication:

A subset S of a vector space V is called **subspace** if

- (a) $\mathbf{0} \in S$.
- (b) $\mathbf{u} + \mathbf{v} \in S$, whenever $\mathbf{u}, \mathbf{v} \in S$.
- (c) $\alpha \mathbf{u} \in S$, whenever $\mathbf{u} \in S$ and α is a scalar.

A subspace is automatically a vector space in its own right, i.e. with addition and scalar multiplication inherited from (coming from) V it satisfies all the 10 axioms.

Example: The set $F = \{(x_1, x_2, x_3); x_1 - 2x_2 + x_3 = 0\}$ is a subspace.

Solution: It is a subspace of \mathbf{R}^3 since if $\mathbf{x}, \mathbf{y} \in S$ then $\mathbf{z} = \mathbf{x} + \mathbf{y}$ satisfy $z_1 - 2z_2 + z_3 = x_1 + y_1 - 2(x_2 + y_2) + x_3 + y_3 = x_1 - 2x_2 + x_3 + y_1 - 2y_2 + y_3 = 0 + 0 = 0$ and $\mathbf{w} = \alpha \mathbf{x}$ satisfy $w_1 - 2w_2 + w_3 = \alpha x_1 - 2\alpha x_2 + \alpha x_3 = \alpha(x_1 - 2x_2 + x_3) = 0$.

Example: The set $G = \{(x_1, x_2, x_3); x_1 - 2x_2 + x_3 = 1\}$ is not a subspace.

Solution: You can see that all three of the conditions fail! Firstly $\mathbf{0}$ is not in the space. If $\mathbf{x}, \mathbf{y} \in G$ then $\mathbf{z} = \mathbf{x} + \mathbf{y} \notin G$ since $z_1 - 2z_2 + z_3 = x_1 + y_1 - 2(x_2 + y_2) + x_3 + y_3 = x_1 - 2x_2 + x_3 + y_1 - 2y_2 + y_3 = 1 + 1 = 2 \neq 1$

Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be vectors in a vector space V . A sum of the form $\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$, where $\alpha_1, \dots, \alpha_n$ are scalars, is called a **linear combination** of $\mathbf{v}_1, \dots, \mathbf{v}_n$. The set of all linear combinations of a $\mathbf{v}_1, \dots, \mathbf{v}_n$ is called the **span** of $\mathbf{v}_1, \dots, \mathbf{v}_n$ and is denoted by $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ **span** (is a **spanning set** for) V if every vector in V can be written as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$.

Ex The plane $x_1 - 2x_2 + x_3 = 0$ is the span of the vectors $(2, 1, 0)^T$ and $(-1, 0, 1)^T$.

Sol $x_1 - 2x_2 + x_3 = 0$ is in reduced row echelon form so x_2 and x_3 are free and the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Th If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are in a vector space V , then $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$ is a subspace of V .

Pf The proof is essentially the same as that the plane $x_1 + x_2 + x_3 = 0$ is a subspace. Let $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$ be an arbitrary element in $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$. then $\beta \mathbf{v} = (\beta \alpha_1) \mathbf{v}_1 + \dots + (\beta \alpha_n) \mathbf{v}_n$ is in V , since its a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$. Similarly if $\mathbf{w} = \beta_1 \mathbf{v}_1 + \dots + \beta_n \mathbf{v}_n$ then $\mathbf{v} + \mathbf{w} = (\alpha_1 + \beta_1) \mathbf{v}_1 + \dots + (\alpha_n + \beta_n) \mathbf{v}_n \in V$.

Example Determine for which values of h the vector $(-4, 3, h)$ lies in the space spanned by $(1, -1, -2)$, $(5, -4, -7)$ and $(-3, 1, 0)$.