MATH 20F. MIDTERM 2. May 23, 2008. Okikiolu

1. Let $\mathcal{B}$ be the basis for $\mathbb{R}^2$ consisting of the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T(\mathbf{b}_1) = 2\mathbf{b}_1 - \mathbf{b}_2, \qquad\qquad T(\mathbf{b}_2) = -\mathbf{b}_1.$$

(a) Write down the matrix $[T]_{\mathcal{B}}$ for T relative to the basis $\mathcal{B}$.

(b). Write down the change of coordinates matrix $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ to go from the coordinates $[\mathbf{x}]_{\mathcal{B}}$ to the **standard** coordinates $\mathbf{x}$.

(c). Compute the matrix $[T]_{\mathcal{E}}$ for T relative to the standard basis $\mathcal{E}$.

2. For each of the matrices below, either diagonalize it, i.e. find a matrix P and a diagonal matrix D such that $A = PDP^{-1}$, or explain why this is not possible.

$$\text{(a)} \quad A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \qquad\qquad \text{(b)} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

3. Suppose that A is an 4×4 matrix whose characteristic polynomial is

$$\det(A - \lambda I) \;=\; (1+\lambda)(1-\lambda)(2-\lambda)^2.$$

For the following statements, decide whether each one is

T. Always true.
F. Always false.
S. Sometimes true (true for some choices of A and false for others).

(a) A is diagonalizable.

(b) The eigenspace corresponding to the eigenvalue 2 has dimension 2.

(c) The rank of $A - 2I$ equals 2.

(d) A is row equivalent to the identity.

(e) The rank of $A - I$ equals 1.

(f) The rank of $A - I$ equals 3.

(g) The rank of $A + 50\,I$ equals 3.

(h) A^T has characteristic polynomial $(1+\lambda)(1-\lambda)(2-\lambda)^2$.

(i) The rank of $A^T - 2I$ equals the rank of $A - 2I$.

(j) The rank of the product $A^T A$ equals 4.

4. Suppose that a dynamical system is described by a difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ where

$$A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A and diagonalize A, that is calculate a matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(b) Write down a formula for the matrix A^k.

(c) Suppose that the dynamical system starts at the vector $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Write down a formula for $\mathbf{x}_k$. What happens as $k \to \infty$?

(d) Suppose that the dynamical system starts at the vector $\mathbf{x}_0 = \begin{bmatrix} a \\ b \end{bmatrix}$. If you know that $\mathbf{x}_k$ eventually approaches $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, what can you say about $\begin{bmatrix} a \\ b \end{bmatrix}$?