MATH 20F. MIDTERM 2. May 23, 2008. Okikiolu

1. (a) $\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.

(c)
$$\begin{tikzcd} \mathbf{x} \arrow[r, "{[T]_{\mathcal{E}}}"] & T(\mathbf{x}) \\ {[\mathbf{x}]_{\mathcal{B}}} \arrow[u, "P_{\mathcal{B}}"] \arrow[r, "{[T]_{\mathcal{B}}}"] & {[T(\mathbf{x})]_{\mathcal{B}}} \arrow[u, "P_{\mathcal{B}}"'] \end{tikzcd},$$

$$[T]_{\mathcal{E}} = P_{\mathcal{B}}\ [T]_{\mathcal{B}}\ P_{\mathcal{B}}^{-1} \ = \ \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \ = \ \begin{bmatrix} 5 & -2 \\ 7 & -3 \end{bmatrix}.$$

2.

$$\text{(a)} \quad \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 & -1 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} \ = \ \lambda(\lambda+2)(2-\lambda).$$

Eigenvalues are $0, -2, 2$. Since these are all different, we know at this point that the matrix is diagonalizable, and

$$D \ = \ \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

To find the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ we solve

$$A \ = \ \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \ \sim \ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \text{so we can take } \mathbf{v}_1 \ = \ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

$$A + 2I \ = \ \begin{bmatrix} 4 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \ \sim \ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \text{so we can take } \mathbf{v}_2 \ = \ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

$$A - 2I \ = \ \begin{bmatrix} 0 & 3 & -1 \\ 0 & -3 & 1 \\ 0 & 1 & -3 \end{bmatrix} \ \sim \ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \text{so we can take } \mathbf{v}_1 \ = \ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

$$\text{(b)} \quad \det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = (1-\lambda)^3.$$

Eigenvalue $\lambda = 1$. But

$$B - I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

The null space of this matrix is clearly not all of $\mathbb{R}^3$ and so there is no basis of eigenvectors of B for $\mathbb{R}^3$. Hence B is not diagonalizable.

3. Suppose that A is an 4×4 matrix whose characteristic polynomial is

$$\det(A - \lambda I) = (1+\lambda)(1-\lambda)(2-\lambda)^2.$$

For the following statements, decide whether each one is

T. Always true.
F. Always false.
S. Sometimes true (true for some choices of A and false for others).

(a) S. (b) S. (c) S.
(Note: (b) and (c) are equivalent since Rank$(A-2I)$+Nullity$(A-2I) = 4$.)
(d) T. (Zero is not an eigenvalue so A is invertible.)
(e) F. (f) T. (g) F (it equals 4).
(h) T. ($\det(A-\lambda I) = \det(A-\lambda I)^T = \det(A^T - \lambda I)$.)
(i) T. (The dimension of the row space equals the dimension of the column space.)
(j) T. (Since A is invertible, so is A^T and hence so is the product.)

4. Eigenvalues 3/2 and 1/2. Eigenvectors

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We can take

$$D = \begin{bmatrix} 3/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

(b) $A^k = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (3/2)^k & 0 \\ 0 & (1/2)^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}.$

(c) $\mathbf{x}_k = A^k \mathbf{x}_0 \; = \; A^k = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (3/2)^k & 0 \\ 0 & (1/2)^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; = \; \frac{1}{2} \begin{bmatrix} (3/2)^k - (1/2)^k \\ (3/2)^k + (1/2)^k \end{bmatrix}.$

Alternatively,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \; = \; \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; - \; \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so

$$\mathbf{x}_k \; = \; \frac{1}{2} \left(\frac{3}{2} \right)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; - \; \frac{1}{2} \left(\frac{1}{2} \right)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

As $k \to \infty$ the vector $\mathbf{x}_k$ is very close to $\frac{1}{2} \left(\frac{3}{2} \right)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Its distance from the origin tends to infinity and it approaches the direction $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(d) If we write $\begin{bmatrix} a \\ b \end{bmatrix} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2$ where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for 3/2 and 1/2 respectively, then

$$\mathbf{x}_k \; = \; A^k \mathbf{x}_0 \; = \; c_1 (3/2)^k \mathbf{v}_1 + c_2 (1/2)^k \mathbf{v}_2.$$

This approaches $\mathbf{O}$ as $k \to \infty$ precisely when $c_1 = 0$, that is when $\begin{bmatrix} a \\ b \end{bmatrix}$ is a multiple of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so $b = -a$.