Lecture 4: Surface area, Three dimensional space.
Example. Calculate the area of a hemisphere of radius R.
Solution. The hemisphere can be constructed as a surface of revolution by revolving the quarter circle

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta, \end{cases} \qquad 0 \le \theta \le \pi/2$$

around the x-axis. Last time we mentioned that if a parametric curve with parameter t, $a \le t \le b$ is revolved around the x-axis, the area of the surface obtained is

$$S = \int_a^b 2\pi y\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt. \tag{10.2.5}$$

In our case it is easy to compute

$$\sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{(-R\sin\theta)^2 + (R\cos\theta)^2} = R.$$

Hence the area is

$$\int_0^{\pi/2} 2\pi\, R\sin\theta\, R\,d\theta = 2\pi R^2 \int_0^{\pi/2} \sin\theta\,d\theta = 2\pi R^2(-\cos\theta)|_0^{\pi/2} = 2\pi R^2.$$

We will explain why the formula for the area of a surface of revolution works. The main point to understand is the formula for the area of a slice of a cone. (See the diagram.) The extreme cases are when the cone is a cylinder or when it is an annulus. The area of the curved part of a cylinder of radius y and width ℓ is

$$2\pi r\ell.$$

The area of an annulus of inner radius y and outer radius $y + \ell$ is

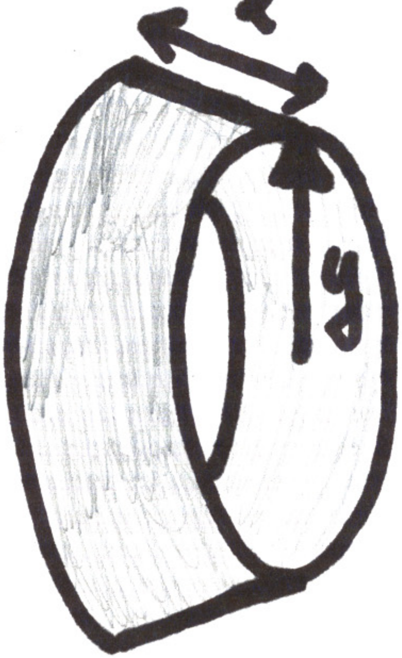
$$\pi(y+\ell)^2 - \pi y^2 = 2\pi y\ell + \pi\ell^2 \approx 2\pi y\ell \qquad \text{if } \ell \text{ is small.}$$

In fact for the general cone, there is a constant α depending on the cone angle so that the area of the slice shown in the diagram is

$$2\pi y\ell + \alpha\pi\ell^2 \approx 2\pi y\ell \qquad \text{if } \ell \text{ is small.}$$

(See 8.2 for the precise derivation.)

Now take points $t_0, t_1, \ldots, t_n$ in the interval $[a, b]$: $a = t_0 < t_1 < ... < t_{i-1} < t_i < ... < t_n = b$ and consider the points $P_i = (x_i, y_i) = (f(t_i), g(t_i))$ on the curve. Let l_i be the length of the line segment L_i joining P_i to P_{i+1}, then revolving L_i around the x-axis gives a slice of cone whose area is approximately $2\pi y_i\ell_i$, and the area of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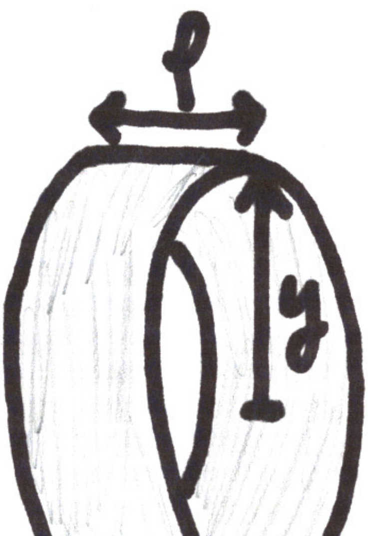
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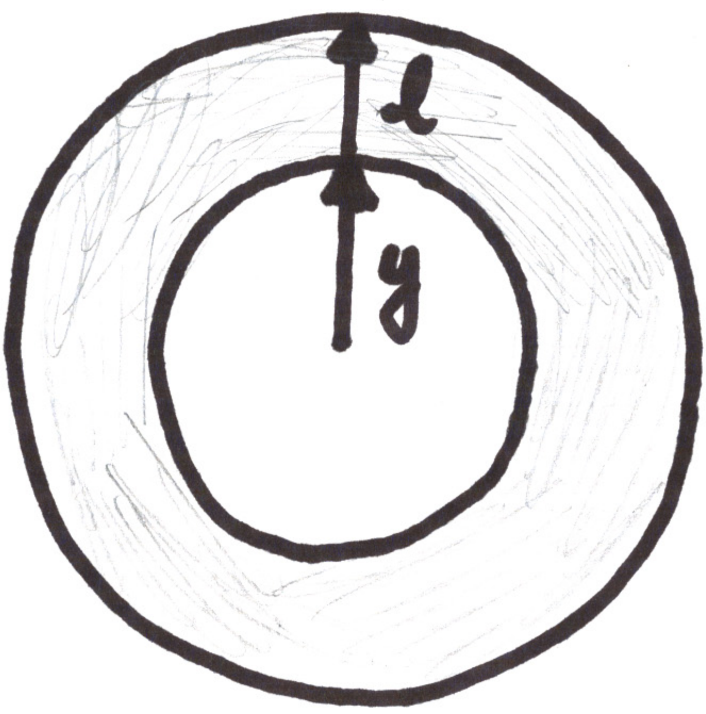
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the whole surface of revolution is approximately the sum over i of these areas. In fact

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi y_i \ell_i,$$

where the number of points, $n+1$, tend to infinity in such a way that the maximum distance $t_i - t_{i-1}$ between neighboring points tends to 0. As in the case of arc length,

$$\ell_i \;=\; \sqrt{(\triangle x_i)^2 + (\triangle y_i)^2} \;\approx\; \sqrt{\left(\frac{dx(t_i)}{dt}\right)^2 + \left(\frac{dy(t_i)}{dt}\right)^2}\,\triangle t_i$$

So

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi y_i \sqrt{\left(\frac{dx(t_i)}{dt}\right)^2 + \left(\frac{dy(t_i)}{dt}\right)^2}\,\triangle t_i$$

and this limit of Riemann sums converges to the integral in (10.2.5).

Three dimensional space.

Coordinates in the plane. Pick a point in the plane for the origin O and draw two perpendicular axes through it. Then there is a correspondence

ordered pairs of real numbers (a, b) $\quad\Leftrightarrow\quad$ points in the plane.

The ordered pair (a, b) corresponds to the point which you get by going a units from the origin in the x-direction and then b units in the y-direction.

There is a convention that the axes are labelled so that if we rotate the positive x-axis about the origin through a quarter turn anticlockwise, we get the positive y-axis.

Coordinates in space. Pick a point in space for the origin O and draw three mutually perpendicular axes through it. Then there is a correspondence

ordered triples of real numbers (a, b) $\quad\Leftrightarrow\quad$ points in space.

The ordered triple (a, b, c) corresponds to the point which you get by going a units from the origin in the x-direction, then b units in the y-direction, then c units in the z-direction.

The convention for labelling the axes is given by the right hand rule: if you stick your thumb in the direction of the positive z-axis, then your fingers curl from the positive x-axis to the positive y-axis in $90°$.

The xy-plane is the plane containing the x and y axes. The yz and xz planes are defined similarly.

Notice that while for the 2-dimensional $x - y$ plane, the points with $x = 0$ form the y-axis, in the 3-dimensional space the points with $x = 0$ form the yz-plane. Similarly, the points with $z = 3$ form a plane 3 units above the xy-plane. In general (although not always), the points (x, y) in the plane satisfying an equation

$f(x, y) = 0$ form a curve, while the points (x, y, z) in space satisfying an equation $f(x, y, z) = 0$ form a surface.

Distance formula In two dimensions the distance between a point $P_1(x_1, y_1)$ and a point $P_2(x_2, y_2)$ is given by the Pythagorean Theorem:

$$|P_1\,P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In three dimensions the distance between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1\,P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

For the proof we note that we are looking for the length d of the diagonal of a rectangular box with edges of length $\triangle x = |x_2 - x_1|$, $\triangle y = |y_2 - y_1|$ and $\triangle z = |z_2 - z_1|$. The length of the diagonal in the two dimensional rectangle with sides $\triangle x$ and $\triangle y$ is by the Pythagorean theorem $\sqrt{(\triangle x)^2 + (\triangle y)^2}$. The diagonal of the box is then also the hypotenuse of a right angle triangle with one side of length $\sqrt{(\triangle x)^2 + (\triangle y)^2}$ and the other side of length $\triangle z$ so by the Pythagorean theorem $d = \sqrt{(\sqrt{(\triangle x)^2 + (\triangle y)^2})^2 + (\triangle z)^2} = \sqrt{(\triangle x)^2 + (\triangle y)^2 + (\triangle z)^2}$.

Example. Find the equation for a sphere with radius R and center at (x_0, y_0, z_0).
Solution. By definition, the sphere is the set of points (x, y, z) at distance R from (x_0, y_0, z_0), that is

$$\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = R.$$

Equivalently,

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.$$

For example, the points (x, y, z) on the sphere with center $(2, 3, -1)$ and radius 4 satisfy the equation

$$(x - 2)^2 + (y - 3)^2 + (z + 1)^2 \;=\; 16.$$