Michael Hansen

Non characteristic surfaces and the Cauchy Problem.

These notes are based on Evans section 4.6.1 and appendix C.1. I start with the method of *straightening out boundaries.*

Definition: Let $U \subseteq \mathbb{R}^n$ be some open region and let $\Gamma = \partial U$. We say Γ is C^k if for each $x^0 \in \Gamma$ there exists an $r > 0$ and a C^k function $\gamma\colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that (upon relabelling and reorienting if necessary) we have,

$$U \cap B(x^0, r) = \{x \in B(x^0, r) \mid x_n > \gamma(x_1, \ldots, x_{n-1})\} \tag{1}$$

Similarly, Γ is said to be C^∞ if C^k for all $k = 1, 2, \ldots$, and Γ is said to be analytic if γ is analytic.

Straightening out the boundary: Define $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ to be,

$$\begin{cases} y_i = x_i := \Phi^i(\mathbf{x}), \quad i = 1, \ldots, n-1 \\ y_n = x_n - \gamma(x_1, \ldots, x_{n-1}) \end{cases} \tag{2}$$

(For short $\mathbf{y} = \Phi(\mathbf{x})$). Similarly, if we put,

$$\begin{cases} x_i = y_i := \Psi^i(\mathbf{y}), \quad i = 1, \ldots, n-1 \\ x_n = y_n + \gamma(y_1, \ldots, y_{n-1}) \end{cases} \tag{3}$$

(again, abbreviated as $\mathbf{x} = \Psi(y)$), then we see $\Phi = \Psi^{-1}$. i.e. Φ straightens out Γ near x^0.

Definition: Let $\nu = (\nu_1, \ldots, \nu_n)$ be the (outward) unit normal of Γ. The j'th normal derivative of u at $x^0 \in \Gamma$ is defined by,

$$\frac{\partial^j u}{\partial \nu^j} = \sum_{|\alpha| = j} (D^\alpha u)\nu^\alpha = \sum_{\alpha_1 + \ldots + \alpha_n = j} \frac{\partial^j u}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} \nu_1^{\alpha_1} \cdots \nu_n^{\alpha_n} \tag{4}$$

Example: $\frac{\partial u}{\partial \nu} = Du \cdot \nu$

Cauchy-problem: We want to solve the k'th order quasi linear PDE,

$$\sum_{|\alpha| = k} a_\alpha(D^{k-1}u, \ldots, u, x) D^\alpha u + a_0(D^{k-1}u, \ldots, u, x) = 0 \tag{5}$$

subject to the boundary conditions (also called the Cauchy data),

$$u = g_0, \quad \frac{\partial u}{\partial \nu} = g_1, \quad \ldots \quad , \frac{\partial^{k-1} u}{\partial \nu^{k-1}} = g_{k-1} \quad \text{on} \quad \Gamma \tag{6}$$

Idea: Seek power series solutions on the form,

$$u = \sum_\alpha u_\alpha x^\alpha \tag{7}$$

Flat boundaries: Let $U = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. i.e. $\Gamma = \partial U = \{x \in \mathbb{R}^n \mid x_n = 0\}$. In this situation $\nu = (0, \ldots, 0, 1)$ and so the Cauchy conditions becomes,

$$u = g_0, \quad \frac{\partial u}{\partial x_n} = g_1, \quad \ldots \quad , \frac{\partial^{k-1} u}{\partial x_n^{k-1}} = g_{k-1} \quad \text{on} \quad \Gamma \tag{8}$$

Question: What further partial derivatives can we compute along Γ? i.e. which partial derivatives of u can we compute in terms of $g_0, \ldots, g_{k-1}$ along Γ?

First, since $u = g_0$ on all of Γ, we can differentiate tangentially, i.e. with respect to x_i ($i = 1, \ldots, n-1$) to find $\frac{\partial u}{\partial x_i} = \frac{\partial g_0}{\partial x_i}$. Since we also have $\frac{\partial u}{\partial x_n} = g_1$ we can determine the full gradient Du along Γ. Similarly we have,

$$\begin{cases} \frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 g_0}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n-1 \\ \frac{\partial^2 u}{\partial x_i \partial x_n} = \frac{\partial g_1}{\partial x_i}, \quad i = 1, \ldots, n-1 \\ \frac{\partial^2 u}{\partial x_n^2} = g_2 \end{cases} \tag{9}$$

Thus we can compute $D^2 u$ along Γ. Continuing to $|\alpha| = 3$ gives,

$$\begin{cases} \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} = \frac{\partial^3 g_0}{\partial x_i \partial x_j \partial x_k}, \quad i, j, k = 1, \ldots, n-1 \\ \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_n} = \frac{\partial^2 g_1}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n-1 \\ \frac{\partial^3 u}{\partial x_i \partial x_n^2} = \frac{\partial g_2}{\partial x_i}, \quad i = 1, \ldots, n-1 \\ \frac{\partial^3 u}{\partial x_n^3} = g_3 \end{cases} \tag{10}$$

and so on. We can continue this and compute $u, Du, \ldots, D^{k-1}u$ along Γ.
However, we run into trouble when we try to compute $D^k u$. We can compute all of $D^k u$ except for $\frac{\partial^k u}{\partial x_n^k}$ since we have no Cauchy data available. However, if the function $a_{(0,\ldots,0,k)} \neq 0$ along γ, we get,

$$\frac{\partial^k u}{\partial x_n^k} = \frac{-1}{a_{(0,\ldots,0,k)}} \left(\sum_{|\alpha|=k, \alpha \neq (0,\ldots,0,k)} a_\alpha D^\alpha u + a_0 \right) \tag{11}$$

Since the rhs can be calculated in terms of the Cauchy data, this allows us to compute all of $D^k u$ along Γ.

Definition: We say $\Gamma = \{x \mid x_n = 0\}$ is non-characteristic for the PDE (**??**) if $a_{(0,\ldots,0,k)} \neq 0$ for all values of its arguments.

How about higher derivatives? We can now extend our list of Cauchy data with $g_k = \frac{\partial^k u}{\partial x_n^k}$ on Γ, and then continue as before and compute all of $D^{k+1}u$ except for the term $\frac{\partial^{k+1} u}{\partial x_n^k}$. But if we differentiate (**??**) with respect to x_n, evaluate the expression on Γ, and then rearrange to find $\frac{\partial^{k+1} u}{\partial x_n^{k+1}} = \frac{1}{a_{(0,\ldots,0,k)}}\{\ldots\}$ where $\{\ldots\}$ denotes something which can be computed in terms of $g_0, \ldots, g_k$ along Γ. Thus we get all of $D^{k+1}u$, and by induction we can now get all partial derivatives of u on the plane Γ.

Definition: We say the surface Γ is non characteristic for the PDE (**??**), provided,

$$\sum_{|\alpha|=k} a_\alpha \nu^\alpha \neq 0 \quad \text{on } \Gamma \tag{12}$$

for all values of the arguments of the coefficients a_k ($|\alpha| = k$).

Theorem: Assume that Γ is non-characteristic for the PDE (**??**). Then if u is a smooth solution of (**??**) and u satisfies the Cauchy conditions,

$$g_0 = u, \quad g_1 = \frac{\partial u}{\partial \nu}, \quad \ldots \quad , \frac{\partial^{k-1} u}{\partial \nu^{k-1}} = g_{k-1} \quad \text{on } \Gamma \tag{13}$$

we can uniquely compute all the partial derivatives of u along Γ in terms of Γ, the functions $g_0, \ldots, g_{k-1}$ and the coefficients a_α ($|\alpha| = k$), a_0.
Proof: The idea is to reduce the problem to the situation with flat boundary.
Therefore, choose $x^0 \in \gamma$ and let Φ, Ψ be the straightening out isomorphisms s.t. $\Phi(\Gamma \cap B(x^0, r)) \subseteq \{y \mid y_n = 0\}$ for some $r > 0$. Define $v(y) = u(\Psi(y))$ so that $u(x) = v(\Psi(x))$. By using $\sum_{|\alpha|=k} a_\alpha D^\alpha u + a_0 = 0$ we get that v satisfies the equation,

$$\sum_{|\alpha|=k} b_\alpha D^\alpha v + b_0 = 0 \tag{14}$$

for some functions b_α, b_0.

Claim: $b_{(0,\ldots,0,k)} \neq 0$ on $y_n = 0$.

Now define the functions $h_0, \ldots, h_{k-1} \colon \mathbb{R}^{n-1} \to \mathbb{R}$ by,

$$v = h_0, \quad \frac{\partial v}{\partial y_n} = h_1, \quad \ldots \quad , \frac{\partial^{k-1} v}{\partial y_n^{k-1}} = h_{k-1} \quad \text{on} \quad \{y_n = 0\} \tag{15}$$

Thus we can compute $h_0, \ldots, h_{k-1}$ near $y = 0$ in terms of Φ and the functions $g_0, \ldots, g_{k-1}$. But using the claim and the case of flat boundary, we see that we can calculate all of the partial derivatives of v on $\{y_n = 0\}$ near $y = 0$. Finally, using $u(x) = v(\Phi(x))$ we observe that we can compute all the partial derivatives of u on Γ near x_0.

Proof of claim: Again we use $u(x) = v(\Phi(x))$ to see that (with $|\alpha| = k$),

$$D^\alpha u = \frac{\partial^k v}{\partial y_n^k} (D\Phi^n)^\alpha + \left\{ \text{terms not involving } \frac{\partial^k v}{\partial y_n^k} \right\} \tag{16}$$

Hence it follows that,

$$\begin{aligned} 0 &= \sum_{|\alpha|=k} a_\alpha D^\alpha u + a_0 \\ &= \sum_{|\alpha|=k} a_\alpha (D\Phi^n)^\alpha \frac{\partial^k v}{\partial y_n^k} + \left\{ \text{terms not involving } \frac{\partial^k v}{\partial y_n^k} \right\} \end{aligned} \tag{17}$$

and so $b_{(0,\ldots,0,k)} = \sum_{|\alpha|=k} a_\alpha \left(D\Phi^n\right)^\alpha$. But $D\Phi^n$ is parallel to ν on Γ. Consequently $b_{(0,\ldots,0,k)}$ is a non-zero multiple of the term,

$$\sum_{|\alpha|=k} a_\alpha \nu^\alpha \neq 0 \tag{18}$$

This verifies the claim.

$\square$