Math 231A Partial Differential Equations

Homework 5

Below are two of the most important measure theory results, formulated for those who may not know about measurable functions etc..

Dominated convergence theorem on $\mathbb{R}^n$. *Suppose that for every $\varepsilon > 0$ we have a continuous (or more generally measurable) function $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}$, and suppose that there exists a continuous (or more generally measurable) function g such that*

$$\int_{\mathbb{R}^n} g(x)\, dx \ < \ \infty,$$

and

$$|f_\varepsilon(x)| \ \leq \ g(x), \qquad\qquad \varepsilon > 0,\ x \in \mathbb{R}^n.$$

Suppose that for each $x \in \mathbb{R}^n$,

$$f_\varepsilon(x) \ \to \ f(x) \qquad\qquad \text{as } \varepsilon \to 0.$$

Then

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} f_\varepsilon(x)\, dx \ = \ \int_{\mathbb{R}^n} f(x)\, dx.$$

Fubini's Theorem on $\mathbb{R}^n$. *Suppose that $f(x, y)$ is continuous on $\mathbb{R}^m \times \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^n} |f(x, y)|\, dydx \ = \ \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |f(x, y)|\, dxdy,$$

with the understanding that both sides can be $+\infty$. If both sides are finite, then

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x, y)\, dydx \ = \ \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f(x, y)\, dxdy.$$

Problem 1. Show that the condition $|f_\varepsilon(x)| \leq g(x)$ in the dominated convergence theorem is necessary, that is exhibit a sequence of continuous functions f_ε on $[0, 1]$ such that $f_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0$ for each $x \in [0, 1]$, but

$$\int_0^1 f_\varepsilon(x)\, dx \ \not\to \ \int_0^1 f(x)\, dx.$$

Problem 2. Suppose that we have a continuous function $g : \mathbb{R}^m \to [0, \infty)$, and a continuous function $f : U \times \mathbb{R}^n \to \mathbb{R}$, where $U \subset \mathbb{R}^m$, such that

$$\int_{\mathbb{R}^n} g(y)\, dy \ < \ \infty,$$

and

$$|f(x,y)| \le g(y), \qquad x \in U,\ y \in \mathbb{R}^n.$$

Define

$$F(x) = \int_{\mathbb{R}^n} f(x,y)\,dy.$$

(a). Prove that F is continuous.

(b). Suppose that $\partial_{x_1} f(x,y)$ exists everywhere. Suppose moreover that

$$|\partial_{x_1} f(x,y)| \le g(y), \qquad x \in U,\ y \in \mathbb{R}^n.$$

Show that $\partial_{x_1} F(x)$ exists at each point $x \in U$.

Problem 3. Suppose that there exists a non-negative function $\phi \in C(\mathbb{R}^n \times [0,\infty))$ such that for each $t > 0$,

$$\int_{\mathbb{R}^n} \phi(x,t)\,dx = 1,$$

for $0 < t_0 < t_1 < \infty$,

$$\int_{\mathbb{R}^n} \sup_{t_0<t<t_1} \phi(x,t)\,dx < \infty,$$

and for each $r > 0$,

$$\int_{|x|>r} \phi(x,t)\,dx \to 0, \qquad \text{as } t \to 0.$$

Suppose that $f \in C(\mathbb{R}^n)$ is bounded, and define

$$u(x,t) = \begin{cases} \int_{\mathbb{R}^n} \phi(x-y,t) f(y)\,dy & t > 0 \\ f(x) & t = 0 \end{cases}.$$

(a). Show that $u \in C(\mathbb{R}^n \times [0,\infty))$.
(b). Show that if $\partial_{x_1} f$ exists and is continuous, then $\partial_{x_1} u$ exists and is in $C(\mathbb{R}^n \times [0,\infty))$.

Problem 4. Suppose that A is a C^1 map from $[0,\infty)$ to the space of $n\times n$ matrices. Show that

$$\partial_t e^{A(t)} = \int_0^1 e^{(1-s)A(t)} A'(t) e^{sA(t)}\,ds.$$

Hint. For any fixed matrix A, the solution to

$$\frac{d}{ds} B(s) = AB(s)$$

is $B(s) = B(0)e^{sA}$. Define

$$B(s,t) = \partial_t e^{sA(t)}.$$

Write down an equation for $\partial B(s,t)/\partial s$ and use Duhamel's principle.