Math 231A Partial Differential Equations

Solutions to 2.5, problem 5, and symmetry of Green's function.

Evans Section 2.5

5. Prove that there is a constant C, depending only on n, such that

$$\max_{B(0,1)} |u| \le C \left(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f| \right)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,1) \\ u = g & \text{on } \partial B(0,1). \end{cases}$$

Solution. We will assume that u is continuous. There are many ways to prove the result, for example using the explicit solution. We opt for a method based on the maximum principle. We will show that if f and g are both non-negative, then the solution u must also be non-negative. Indeed, using our favorite formula

$$\frac{d}{dr} \fint_{\partial B(x,r)} u(y)\, dS(y) \;=\; \frac{r}{n} \fint_{B(x,r)} \Delta u(y)\, dy,$$

we see that the set of points $x \in B(0,1)$ where u attains its miminum is open. However this set is also closed by continuity of u and so it is either empty or the whole ball $B(0,1)$. That is, u either attains its minimum on $\partial B(0,1)$, in which case this minimum equals $\min g \ge 0$, or u is constant. Now consider the function

$$v(x) \;=\; \frac{\max f}{2n} |x|^2 \;+\; \max g \;-\; u.$$

Then

$$\begin{cases} -\Delta v = (\max f) - f & \text{in } B^0(0,1) \\ u = (\max g) - g & \text{on } \partial B(0,1). \end{cases}$$

Since the functions $(\max f) - f$ and $(\max g) - g$ are positive, we conclude $v \ge 0$, so

$$u(x) \;\le\; \frac{\max f}{2n} |x|^2 \;+\; \max g \;\le\; \frac{\max f}{2n} \;+\; \max g.$$

Similarly, by showing that

$$w(x) \;=\; u(x) \;-\; \frac{(\min f)}{2n} |x|^2 \;-\; (\min g),$$

we conclude

$$\frac{\min f}{2n} \;+\; \min g \;\le\; u,$$

so

$$|u| \;\le\; \frac{\max|f|}{2n} \;+\; \max|g| \;\le\; \max|f| + \max|g|.$$

Symmetry of Green's function. $U \subset \mathbb{R}^n$ is open and bounded with smooth (i.e. C^∞) boundary. Suppose that for $x, y \in \bar{U}$,

$$G(x,y) \;=\; \Phi(x-y) \;-\; \phi(x,y),$$

where

$$\Phi(x) \;=\; \frac{1}{(n-2)|\partial B(0,1)|\,|x|^{n-2}},$$

and suppose $\phi(x,y)$ is smooth on the set

$$\bar{U} \times \bar{U} \;\setminus\; \{(x,y) : x, y \in \partial U, \quad x = y\},$$

and satisfies

$$(1) \qquad \begin{cases} \Delta_y \phi(x,y) = 0, & x \in \bar{U}, \quad y \in U, \\ \phi(x,y) = \Phi(x-y), & x \in \bar{U}, \quad y \in \partial U, \quad y \neq x. \end{cases}$$

By considering the integral
(*)

$$\int_{U\setminus(B(x,\epsilon)\cup B(y,\epsilon))} G(x,y)\Delta_y G(z,y)\,dy \;-\; \int_{U\setminus(B(x,\epsilon)\cup B(y,\epsilon))} (\Delta_y G(x,y))G(z,y)\,dy,$$

show that for $x, z \in U$,

$$G(x,z) = G(z,x).$$

Proof. Notice that the integrands in (*) vanish, because the domain of integration does not include the points x and z where the Green's functions blow up. Applying Green's Theorem to these integrals and splitting up the boundary of the region of integration into its three components, we conclude

$$\begin{aligned} 0 \;=\; & \int_{\partial U} G(x,y)\frac{\partial}{\partial \nu_y} G(z,y)\,dS(y) \;-\; \int_{\partial U} G(z,y)\frac{\partial}{\partial \nu_y} G(x,y)\,dS(y) \\ & + \int_{\partial B(x,\epsilon)} G(x,y)\frac{\partial}{\partial \nu_y} G(z,y)\,dS(y) \;-\; \int_{\partial B(x,\epsilon)} G(z,y)\frac{\partial}{\partial \nu_y} G(x,y)\,dS(y) \\ (**) \qquad & \\ & + \int_{\partial B(z,\epsilon)} G(x,y)\frac{\partial}{\partial \nu_y} G(z,y)\,dS(y) \;-\; \int_{\partial B(z,\epsilon)} G(z,y)\frac{\partial}{\partial \nu_y} G(x,y)\,dS(y). \end{aligned}$$

Because the region of integration is outside the ball $B(x,\epsilon)$, the outward normal ν_g on $\partial B(x,\epsilon)$ is directed in toward the center of the ball $B(x,\epsilon)$, and similarly for x replaced by z.

Now the terms on the first line of (**) both vanish. Indeed, $G(z,y)$ vanishes for $z \in U$ and $y \in \partial U$ by the boundary condition (1), and since $x \in U$ which does not intersect the compact set ∂U, we see that $G(x,y)$ is a smooth function of y for y in a neighborhood of ∂U, hence $\partial G(x,y)/\partial \nu_y$ is bounded on ∂U. Hence the integrands in the first line of (**) equal zero. We consider what happens to the other integrals as $\epsilon \to 0$. Note that ϕ is smooth for $x, y, z \in U$. Furthermore, when ϵ is less than the distance from x to z, we see that $G(z,y)$ is smooth on $\overline{B(x,\epsilon)}$ and $G(x,y)$ is smooth on $\overline{B(z,\epsilon)}$. Hence

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} G(z,y)\frac{\partial}{\partial \nu_y}\phi(x,y)\,dS(y) &= 0. \\
\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} G(z,y)\frac{\partial}{\partial \nu_y}\Phi(x,y)\,dS(y) &= G(z,x). \\
\lim_{\epsilon\to 0}\int_{\partial B(z,\epsilon)} G(z,y)\frac{\partial}{\partial \nu_y}\phi(x,y)\,dS(y) &= 0, \\
\lim_{\epsilon\to 0}\int_{\partial B(z,\epsilon)} G(z,y)\frac{\partial}{\partial \nu_y}\Phi(x,y)\,dS(y) &= 0,
\end{aligned}$$

Summing up we get

$$\lim_{\epsilon\to 0}\left(\int_{\partial B(z,\epsilon)} G(x,y)\frac{\partial}{\partial \nu_y}G(z,y)\,dS(y) - \int_{\partial B(z,\epsilon)} G(z,y)\frac{\partial}{\partial \nu_y}G(x,y)\,dS(y)\right) = G(z,x).$$

Similarly this holds with x and z interchanged, so taking the limit in (**) as $\epsilon \to 0$ gives $G(z,x) = G(x,z)$.