MATH 231A PARTIAL DIFFERENTIAL EQUATIONS

Lecture 1: The Vibrating String.

Suppose an elastic string is stretched between the points $x = 0$ and $x = \ell$ on the x-axis, so that there is a tension in the string, and it is allowed to vibrate in the $x - y$ plane. We are going to derive the equation which describes the vibrations. We will start by making two simplifying assumptions.

1. Assume that the only force acting on the string is the tension within the string and this acts tangent to the string. We assume then that there is no gravity, and the frictional forces on the string can be ignored.

2. Assume that each point of the string moves only in the y direction with the x-value fixed. Hence we can describe the motion of the string by a single function $y = u(x, t)$ where t is time. This will generally not be the case, but for small vibrations around the rest state it is approximately true.

We apply Newton's law to the portion of the string between $x = x_0$ and $x = x_1$. Denoting the magnitude of the tension in the string by $T(x, t)$, and writing u_x for the partial derivative of u with respect to x, we have that the force on this piece of string in the y-direction is equal to mass times acceleration.

$$T(x_1,t)\frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}} \;-\; T(x_0,t)\frac{u_x(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} \;=\; \int_{x=x_0}^{x_1} m(x,t)u_{tt}(x,t)\,dx.$$

Differentiating with respect to x_1, we get

$$\left(T(x,t)\frac{u_x(x,t)}{\sqrt{1+u_x^2(x,t)}}\right)_x \;=\; m(x,t)u_{tt}(x,t).$$

If the oscillations are small, we can assume $|u_x| << 1$ and T and m are approximately constant, and we get the wave equation

$$Tu_{xx} = mu_{tt}.$$

This is the classical wave equation with one space dimension. To make the exposition easier, we can choose units so $\ell = \pi$ and $T = m = 1$. We are interested then in the equation

$$(*) \qquad \begin{cases} u_{xx} = u_{tt} & x \in [0,\pi], \quad t \geq 0 \\ u(x,0) = g(x) & x \in [0,\pi] \\ u_t(x,0) = h(x) & x \in [0,\pi] \\ u(0,t) = u(\pi,t) = 0 & t \geq 0. \end{cases}$$

Later on we will solve this by factorizing

$$(\partial_x^2 - \partial_t^2)u \;=\; (\partial_x - \partial_t)(\partial_x + \partial_t)u,$$

and solving two first order equations. For now, however, we use a method which turns out to be meaningful for the vibrating string and which can be applied to compare the wave equation with other equations

Sine Series: We expand a function f on $[0,\pi]$ in a series

$$f(x) \;=\; \sum_{j=1}^{\infty} \hat{f}_j \sin jx.$$

Now the coefficients $\hat{f}_j$ can be calculated by the orthogonality relation

$$\int_0^{\pi} \sin jx \quad \sin kx\, dx \;=\; \begin{cases} 0 & j \neq k \\ \pi/2 & j = k. \end{cases}$$

Hence we must have

$$\hat{f}_j \;=\; \frac{2}{\pi} \int_0^{\pi} f(x) \sin jx\, dx.$$

The Sine series can be a good expansion to make when $f(0) = f(\pi) = 0$. (The functions $\sin jx$ are actually the functions one obtains by “separating variables” in the equation (*).)

Now we suppose that $u(x,t)$ is a solution to (*) and expand it as a Fourier series in x,

$$u(x,t) \;=\; \sum_{j=1}^{\infty} u_j(t) \sin jx. \tag{1}$$

Plugging this into (*), we get

$$\sum_{j=1}^{\infty} (\ddot{u}_j(t) + j^2 u_j(t)) \sin jx \;=\; 0.$$

Hence for each j,

$$\ddot{u}_j(t) \;+\; j^2 u_j(t) \;=\; 0.$$

Factorizing over the complex numbers, we have

$$(d_t - ij)(d_t + ij) u_j(t) \;=\; 0,$$

and the general complex solution is $\alpha_j e^{ijt} + \beta_j e^{-ijt}$ and the general real solution is

$$u_j(t) \;=\; a_j \cos jt \;+\; b_j \sin jt.$$

Hence we get

$$u(x,t) \;=\; \sum_{j=1}^{\infty} (a_j \cos jt + b_j \sin jt) \sin jx.$$

Pluging in the initial conditions from (*), we get

$$(**) \qquad u(x,t) \;=\; \sum_{j=1}^{\infty} \left(\hat{g}_j \cos jt + \frac{\hat{h}_j}{j} \sin jt \right) \sin jx.$$

We thus arrive at an expression for the solution. It is physically significant because it expresses the vibrations of the string as a superposition of vibrations of different frequencies. The frequencies predicted by this model are in very good agreement with experiment, although in real life there are frictional forces which cause the vibrations to be damped down.

Technical Questions.

Question 1. Existence: If we start with smooth enough functions g and h with $g(0) = g(\pi) = 0$, does (**) give a solution to (*)?
Question 2. Uniqueness: If a solution to (*) exists, must it be given by (**)?

The answer to the second question is “yes”. Indeed, if f is twice differentiable with $f(0) = f(\pi) = 0$, then the Sine series of f converges, and f is equal to its Sine series. Hence if $u(x,t)$ is a twice differentiable solution to (*), then indeed (1) holds with

$$u_j(t) \;=\; \frac{2}{\pi} \int_0^{\pi} u(x,t) \sin jx \, dx.$$

The rest of the derivation goes through and we conclude (**).

The first question is a bit more tricky. It turns out to be easy to see that the solution (**) is differentiable, but without doing some work it is only obvious that the second derivatives of (**) are square integrable - the continuity properties are unclear. In reality there may actually be discontinuities in the second derivatives of u. These are occur when the second derivative of h does not vanish at $x = 0$ or $x = \pi$. For the interested reader, we state a little theory.

The Fourier Sine coefficients of f can fail to decay fast for two reasons, either because f is not differentiable enough times, or because some even order derivatives of f fail to vanish at the endpoints.

Theorem. *If* $\sum |a_j| < \infty$ *then the series*

$$a(x) \;:=\; \sum_{j=1}^{\infty} a_j \sin jx$$

converges to a continuous function which vanishes at $x = 0$ *and* $x = \pi$*.*

If $\sum |ja_j| < \infty$ *then* $a(x)$ *is continuously differentiable and*

$$a'(x) \;=\; \sum_{j=1}^{\infty} ja_j \cos jx.$$

Similarly if $\sum |j^2 a_j| < \infty$ *then* $a(x)$ *is twice continuously differentiable and*

$$a''(x) \;=\; -\sum_{j=1}^{\infty} j^2 a_j \sin jx.$$

However, we see that in this case $a''(0) = a''(\pi) = 0$.

Theorem. *If* f *is in* $C^k([0,\pi])$ *(*k *times continuously differentiable) with* $k = 0, 1$ *or* 2*, and* $f(0) = f(\pi) = 0$*, then the Fourier coefficients* $\hat{f}_j$ *satisfy*

$$\sum_j |j^k \hat{f}_j|^2 < \infty. \tag{2}$$

The converse may not be true, that is if (2) holds, it may not be true that $f \in C^k$*. If* $k = 0$ *we can conclude* $f \in L^2$*. If* $k = 1, 2$ *we can conclude* $f \in C^{k-1}$*, and* f *has a* k*th derivative which lies in* L^2*.*

Further Reading

Steven J. Cox, Aye, There's the rub. An inquiry into why a plucked string comes to rest. *Six Themes on Variation.* AMS Student Math Library **26** (2004), 37–57.

Yitzhak Katznelson, *An introduction to harmonic analysis.* Cambridge Mathematical Library, 2004.