Math 231A Partial Differential Equations

Lecture 10: Uniqueness

Last Time: If $g \in C(\partial B^n)$, then the smooth function

$$u(x) \;=\; (1-|x|^2) \fint \frac{g(y)}{|x-y|^n}\, dS(y)$$

solves

$$\begin{cases} \Delta u \;=\; 0, & \text{in } B^n \\ u(x) \to g(x_0), & \text{as } x \to x_0,\;\; x \in B^n,\; x_0 \in \partial B^n. \end{cases}$$

Note: defining $u(x) = g(x)$ for $x \in \partial B^n$, then $u \in C(\overline{B^n}) \cap C^\infty(B^n)$ solves

$$\begin{cases} \Delta u = 0 & \text{in } B^n \\ u = g & \text{on } \partial B^n. \end{cases}$$

Now we will see that this is the unique solution.

Uniqueness Theorem. *If U is open and bounded in $\mathbb{R}^n$ and ∂U is C^1, then there exists at most one function $u \in C^2(\bar{U})$ solving*

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

Proof. If $\tilde{u}$ is another solution, set $w = u - \tilde{u}$, then

$$\begin{cases} \Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U. \end{cases}$$

$$\int_U w \Delta w \, dx \;=\; -\int_U |\nabla w|^2 \, dx.$$

Hence $\nabla w = 0$ in U, and so w is constant. Since $w = 0$ on ∂U, $w = 0$ on U.

This result is interesting because it involves the *energy* $\int_U |\nabla w|^2 \, dx$. However, it is not quite satisfactory because it requires $u \in C^2(\bar{U})$. We get rid of this assumption by a different proof using the *maximum principle.*

Strong Maximum Principle. . *Suppose $U \subset \mathbb{R}^n$ is open and connected, and $u \in C^2(U)$ is harmonic on U. If there exists $x_0 \in U$ such that*

$$u(x) \;\leq\; u(x_0)$$

for all $x \in U$, then u is constant on U.

Maximum Principle *If U is open and bounded in $\mathbb{R}^n$ and $u \in C(\bar{U}) \cap C^2(U)$ is harmonic in U, then u attains its maximum on ∂U.*

Proof of SMP. Consider

$$V \;=\; \{x \in U : u(x) = u(x_0)\}.$$

Then V is closed in U because u is continuous. But now suppose $x \in V$. Then by the mean value theorem, if $\overline{B(x,r)} \subset U$,

$$u(x) \;=\; ⨍_{B(x,r)} u(y)\, dy,$$

But $u(y) \leq u(x)$ on $B(x,r)$, so we must have $u(y) = u(x) = u(x_0)$ on $B(x_0,r)$. Hence V is open and closed and since U is connected, $U = V$.

Proof of MP. Since u is continuous on the compact set $\bar{U}$, u attains its maximum on $\bar{U}$. Suppose that u attains its maximum at $x_0 \in U$. Then by the previous result, u is constantly equal to its maximum on the component of U containing x_0, and hence at boundary points of that component.

Uniqueness. . *If U is open and bounded in $\mathbb{R}^n$, then there exists at most one function $u \in C^2(U) \cap C(\bar{U})$ solving*

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

Proof. If $\tilde{u}$ is another solution, then $u - \tilde{u}$ is harmonic on U and hence attains its maximum on ∂U, but it vanishes on ∂U, so $u - \tilde{u} \leq 0$. Reversing the roles of u and $\tilde{u}$, we see that $\tilde{u} - u \leq 0$, so $u = \tilde{u}$.

Corollary. *If $u \in C(B^n) \cap C^2(B^n)$ is harmonic in B^n, then for $x \in B^n$,*

$$u(x) \;=\; (1 - |x|^2) ⨍_{\partial B^n} \frac{u(y)}{|x-y|^n}\, dS(y).$$

This follows immediately from the uniqueness result. We also see that if $u \in C(\partial B(x,r)) \cap C^2(B(x,r)$ is harmonic in $B(x,r)$, then since $z \to u(x+rz)$ is harmonic on B^n,

$$u(x+rz) \;=\; (1 - |x|^2) ⨍_{\partial B^n} \frac{u(x+ry)}{|x-y|^n}\, dS(y),$$

so substituting $X = x + rz$ and $Y = x + ry$, we have

$$(*) \quad u(X) \;=\; (1 - |X-x|^2/r|^2) ⨍_{\partial B(x,r)} \frac{u(Y)}{|(X-Y)/r|^n}\, dS(Y)$$
$$= \; r^{n-2}(r^2 - |X-x|^2) ⨍_{\partial B(x,r)} \frac{u(Y)}{|X-Y|^n}\, dS(Y).$$

Corollary. *If U is open in $\mathbb{R}^n$ and u is harmonic on U, then u is smooth on u.*

We remark that there are other ways to prove this than by using the explicit representation formula. Indeed suppose $\rho \in C_c^\infty(\mathbb{R}^n)$ is rotationally invariant (radial) and $\int_{\mathbb{R}^n} \rho \, dx = 1$, and suppose that the support of ρ is contained in $B(0, \varepsilon)$. Now let u be a harmonic function on U, and suppose that $x \in U$ is more than distance ε from the boundary of U. Then using the mean value theorem, we find that on a neighborhood of x, $u = \rho * u$. Hence u is smooth. Details are given in Evans 2.2.3b.

We can differentiate (*) to estimate the derivatives of u.

Proposition. *If $u \in C(\partial B(x,r))$ is harmonic in $B(x,r)$, then*

$$|\nabla u(x)| \;\leq\; \left(\sup_{B(0,r)} |u| \right) \, \frac{n+2}{r}.$$

Liouville's Theorem. . *If u is harmonic and bounded on $\mathbb{R}^n$ then u is constant.*

Proof. By the previous propositon, taking $r \to \infty$ we see that ∇u vanishes at each point $x \in \mathbb{R}^n$, and hence u is constant.

Theorem. . *If $u \in C^2(\mathbb{R}^n)$ is bounded and satisfies $-\Delta u = f$, then*

$$u \;=\; \Phi * f \;+\; C$$

for some constant C.

Proof. Set $\tilde{u} = \Phi * f$, so $-\Delta \tilde{u} = f$ and $\tilde{u}$ is bounded. Then $w = u - \tilde{u}$ is harmonic and bounded on $\mathbb{R}^n$. Hence by Liouville's theorem w is constant.