Math 231A Partial Differential Equations

Lecture 11: Fourier Transform

For $u : \mathbb{R}^n \to \mathbb{C}$,

$$\|u\|_p \;=\; \left(\int_{\mathbb{R}^n} |u|^p \, dx\right)^{1/p}, \qquad\qquad \|u\|_\infty \;=\; \operatorname{ess\,supp}_{\mathbb{R}^n} |u| \;=\; \inf_{|V|=0} \; \sup_{\mathbb{R}^n \setminus V} |u|.$$

Then for $p < \infty$, L^p is the completion of $C(\mathbb{R}^n)$ in the norm $\| \; \|_p$. It is also the space of measurable functions for which the norm $\| \; \|_p$ is finite.

For $u \in L^1(\mathbb{R}^n)$, define

$$\begin{aligned} \hat{u}(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot y} \, dx, \\ \check{u}(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot y} \, dx \;=\; \hat{u}(-x). \end{aligned}$$

Propositon.
(a). If $v, w \in L^1(\mathbb{R}^n)$, then $\hat{v}, \hat{w} \in L^\infty(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} v\hat{w} \, dx \;=\; \int_{\mathbb{R}^n} \hat{v} w \, dx.$$

(b). If $\phi(x) = e^{-t|x|^2}$, then $\hat{\phi}(x) = (1/2t)^{n/2} e^{|x|^2/4t}$.

(c). (i) If $u, v \in L^1(\mathbb{R}^n)$, then

$$\widehat{(u * v)} = (2\pi)^{n/2} \hat{u}\hat{v}.$$

(ii) If $u, v \in L^2(\mathbb{R}^n)$, then $u * v$ is continuous.

(d). If $u \in L^1 \cap L^2$, then

$$\|u\|_2 \;=\; \|\hat{u}\|_2 \;=\; \|\check{u}\|_2.$$

Hence the Fourier transform extends to define an isometry of $L^2(\mathbb{R}^n)$.

(e). If $u \in L^2(\mathbb{R}^n)$, then $\check{\hat{u}} = u$.

(f). If $u, D^\alpha u \in L^2(\mathbb{R}^n)$, then $\widehat{D^\alpha u}(x) \;=\; (ix)^\alpha \hat{u}(x)$.

What interests us most is (f) which shows that the Fourier transform diagonalizes the operation of differentiation. The theory is slightly more technical than that of Fourier series because the "eigenfunctions" $\phi_y(x) = e^{ix\cdot y}$ are not in the Hilbert space $L^2(\mathbb{R}^n)$.

Proof of the Theorem.

(a). By Fubini's Theorem,

$$\begin{aligned}\int_{\mathbb{R}^n} v\hat{w}\,dx &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} v(x)\int_{\mathbb{R}^n} w(y)e^{-ix\cdot y}\,dydx \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} v(x)w(y)e^{-ix\cdot y}\,dxdy \;=\; \int_{\mathbb{R}^n}\hat{v}w\,dy.\end{aligned}$$

(b). In one dimension,

$$\begin{aligned}\hat{\phi}(y) &= \frac{1}{(2\pi t)^{1/2}}\int_{-\infty}^{\infty} e^{-tx^2}e^{-ixy}\,dx \\ &= \frac{1}{(2\pi t)^{1/2}}\int_{-\infty}^{\infty} e^{-x^2}e^{-ixy/t^{1/2}}\,dx \\ &= \frac{e^{-|y|^2/4t}}{(2\pi t)^{1/2}}\int_{-\infty}^{\infty} e^{-(x-iy/(2t^{1/2}))^2}\,dx \\ &= \frac{e^{-|y|^2/4t}}{(2\pi t)^{1/2}}\int_{\Gamma} e^{-z^2}\,dz,\end{aligned}$$

where Γ is the line in the complex plane $\Im z = y/(2t^{1/2})$ oriented from $\Re z = -\infty$ to $\Re z = \infty$. We claim that this contour can be deformed to the real line. Indeed, consider the rectangle R given by $0 \le \Im z \le y/(2t^{1/2})$, and $|\Re z| \le N$. Then since e^{-z^2} is analytic on R, $\int_{\partial R} e^{-z^2}\,dz = 0$. But the integral along the sides parallel to the imaginary axis tends to zero as $N \to \infty$, since when $z = \pm N + i\tau$, we have $|e^{-z^2}| = e^{-N^2+\tau^2} \to 0$ as $N \to \infty$, uniformly on $0 \le \tau \le y/(2t^{1/2})$. However,

$$\int_{-\infty}^{\infty} e^{-z^2}\,dz = \sqrt{\int_{\mathbb{R}^2} e^{-x^2}e^{-y^2}\,dxdy} = \sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-r^2}r\,drd\theta} = \sqrt{\pi}.$$

This completes the proof of the case $n = 1$. For $n > 1$, we have

$$\begin{aligned}\hat{\phi}(y) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-t|x|^2}e^{-ix\cdot y}\,dx &= \prod_{j=1}^{n}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-t|x_j|^2}e^{-ix_jy_j}\,dx_j \\ &= \prod_{j=1}^{n}\frac{1}{(2t)^{1/2}}e^{-|y_j|^2/4t} = \frac{1}{(2t)^{n/2}}e^{-|y|^2/4t}.\end{aligned}$$

so the result follows from the one dimensional case.

(c). Note that $u * v \in L^1$. Indeed,

$$\int_{\mathbb{R}^n} |u*v| = \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} u(x-y)v(y)\,dy\right|\,dx \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x-y)||v(y)|\,dxdy = \|u\|_1\,\|v\|_1.$$

We have

$$\widehat{u*v}(x) \;=\; \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} u{*}v(y)e^{-ix\cdot y}\,dy \;=\; \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(y{-}z)v(z)e^{-ix\cdot y}\,dzdy$$

$$= \;\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(y-z)e^{-ix\cdot(y-z)}v(z)e^{-ix\cdot z}\,dydz \;=\; (2\pi)^{n/2}\hat{u}(x)\hat{v}(x).$$

Now we suppose $u \in L^2(\mathbb{R}^n)$. Write $v^{(h)}(x) = v(x-h)$. Then

$$|u*v(x+h) - u*v(x)| \;=\; \left|\int_{\mathbb{R}^n}(u(x+h-y) - u(x-y))v(y)\,dy\right| \;\le\; \|u^{(h)}{-}u\|_2\|v\|_2.$$

However, $\|u^{(h)} - u\|_2 \to 0$ as $h \to 0$. Indeed, given $\varepsilon > 0$, choose $w \in C_c(\mathbb{R}^n)$ with $\|u - w\|_2 < \varepsilon$. Choose N so that the support of w is contained in $B(0, N-1)$. But w is uniformly continuous, so there exists δ with $0 < \delta < 1$ such that $\|w^{(h)} - w\|_\infty < \varepsilon/\sqrt{|B(0,N)|}$ for $|h| < \delta$. But w and $w^{(h)}$ are supported in $B(0,N)$, so $\|w^{(h)} - w\|_2 < \varepsilon$ for $|h| < \delta$. But then when $|h| < \delta$,

$$\|u^{(h)} - u\|_2 \;\le\; \|u^{(h)} - w^{(h)}\|_2 \;+\; \|w^{(h)} - w\|_2 \;+\; \|w - u\|_2 \;\le\; 3\varepsilon.$$

(d). Set $v(x) = \bar{u}(-x)$. Then consider

$$w \;=\; u*v.$$

Since $u, v \in L^2(\mathbb{R}^n)$, we see that w is continuous. Moreover,

$$w(0) \;=\; \int_{\mathbb{R}^n} u(-y)\bar{u}(-y)\,dy \;=\; \int_{\mathbb{R}^n} u\bar{u}\,dy \;=\; \|u\|_2^2.$$

But

$$\hat{v}(y) \;=\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\bar{u}(-x)e^{-ix\cdot y}\,dx \;=\; \frac{1}{(2\pi)^n}\overline{\int_{\mathbb{R}^n} u(-x)e^{ix\cdot y}\,dx}$$

$$= \;\frac{1}{(2\pi)^n}\overline{\int_{\mathbb{R}^n} u(x)e^{-ix\cdot y}\,dx} \;=\; \overline{\hat{u}(y)}.$$

Hence $\hat{w} = (2\pi)^{n/2}|\hat{u}|^2$, and

$$\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\hat{w}\,dx \;=\; \int_{\mathbb{R}^n}|\hat{u}|^2\,dx \;=\; \|\hat{u}\|_2^2.$$

Hence we want to show that

$$w(0) \;=\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\hat{w}\,dx.$$

The right hand side is just $\check{\hat{w}}$, so we want to show the Fourier inversion formula for the function w at 0, namely

$$w(0) \;=\; \check{\hat{w}}(0).$$

We will finish this this next time.