Lecture 12: Fourier Inversion formula

Proposition.
(a). If $v, w \in L^1(\mathbb{R}^n)$, then $\hat{v}, \hat{w} \in L^\infty(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} v\hat{w}\,dx \;=\; \int_{\mathbb{R}^n} \hat{v}w\,dx.$$

(b). If $\phi(x) = e^{-t|x|^2}$, then $\hat{\phi}(x) = (1/2t)^{n/2}e^{|x|^2/4t}$.

(c). (i) If $u, v \in L^1(\mathbb{R}^n)$, then

$$\widehat{(u * v)} = (2\pi)^{n/2}\hat{u}\hat{v}.$$

(ii) If $u, v \in L^2(\mathbb{R}^n)$, then $u * v$ is bounded and continuous.

(d). If $u \in L^1 \cap L^2$, then

$$\|u\|_2 \;=\; \|\hat{u}\|_2 \;=\; \|\check{u}\|_2.$$

Hence the Fourier transform extends to define an isometry of $L^2(\mathbb{R}^n)$.

(e). If $u \in L^2(\mathbb{R}^n)$, then $\check{\hat{u}} = u$.

(f). If $u, D^\alpha u \in L^2(\mathbb{R}^n)$, then $\widehat{D^\alpha u}(x) \;=\; (ix)^\alpha \hat{u}(x)$.

Last time, we proved (a)-(c), and we started the proof of (d). Before proceeding, we will need the following results regarding approximate identities.

Proposition. *Suppose that we have a family of positive functions $\phi_\varepsilon : \mathbb{R}^n \to [0, \infty)$ such that for each $\varepsilon > 0$, such that for each $\varepsilon > 0$,*

$$\int_{\mathbb{R}^n} \phi_\varepsilon \;=\; 1,$$

for each $r > 0$,

$$\int_{|x|>r} \phi_\varepsilon \;\to\; 0 \qquad \text{as} \quad \varepsilon \to 0.$$

(A). If $L^\infty(\mathbb{R}^n)$ is continuous at $x = 0$, then

$$\int_{\mathbb{R}^n} f\phi_\varepsilon \;\to\; f(0) \qquad \text{as} \quad \varepsilon \to 0.$$

(B). If $f \in L^2(\mathbb{R}^n)$, then

$$\phi_\varepsilon * f \;\to\; f \qquad \text{in} \quad L^2(\mathbb{R}^n), \qquad \text{as} \quad \varepsilon \to 0.$$

(This is also true for 2 replaced by $p \geq 1$, although we will not need this.)

Proof of the Proposition. (A).

$$(*) \quad \left| \int_{\mathbb{R}^n} f\phi_\varepsilon \ - \ f(0) \right| \ = \ \left| \int_{\mathbb{R}^n} (f(x) - f(0))\phi_\varepsilon(x)\, dx \right|$$

$$\leq \int_{|x|<r} |f(x) - f(0)|\phi_\varepsilon(x)\, dx \ + \ \int_{|x|>r} |f(x) - f(0)|\phi_\varepsilon(x)\, dx$$

$$\leq \sup_{|x|<r} |f(x) - f(0)| \ + \ 2\|f\|_\infty \int_{|x|>r} \phi_\varepsilon(x)$$

Given $\delta > 0$, there exists $r > 0$ such that $|f(x) - f(0)| < \delta$ when $|x| < r$. But then there exists $\varepsilon_0 > 0$ such that $\int_{|x|<r} \phi_\varepsilon < \delta/(2\|f\|_\infty)$ when $\varepsilon < \varepsilon_0$. Hence when $\varepsilon < \varepsilon_0$, the right hand of (*) is less than 2δ. This proves (A).

(B). First note that we can compute the L^2 norm of a function w by

$$\|w\|_2 \ = \ \sup_{\|h\|_2 \leq 1} \int_{\mathbb{R}^n} |wh|.$$

This can be used in particular to show that $L^2 * L^1 \subset L^2$. Indeed,

$$\int_{\mathbb{R}^n} |f * g||h| \ \leq \ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x - y)g(y)h(x)|\, dydx$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x - y)g(y)h(x)|\, dxdy \ \leq \ \|f\|_2\|h\|_2 \int_{\mathbb{R}^n} |g| = \|f\|_2\|h\|_2\|g\|_1.$$

Hence we get

$$\|f * g\|_2 \ \leq \ \|f\|_2\|g\|_1.$$

Setting $f^{(y)}(x) = f(x - y)$ and applying a similar argument we get

$$\int_{\mathbb{R}^n} |f * \phi_\varepsilon - f||h| \ = \ \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (f(x - y) - f(x))\phi_\varepsilon(y)\, dy \right| |h(x)|\, dx$$

$$\leq \int_{\mathbb{R}^n} \int_{|y|<r} |f(x-y)-f(x)||h(x)||\phi_\varepsilon(y)|\, dxdy + \int_{\mathbb{R}^n} \int_{|y|>r} |f(x-y)-f(x)||h(x)||\phi_\varepsilon(y)|\, dxdy$$

$$\leq \sup_{|y|<r} \|f^{(y)} - f\|_2\|h\|_2 \ + \ 2\|f\|_2\|h\|_2 \int_{|y|>r} \phi_\varepsilon(y)\, dy.$$

Now given $\delta > 0$, we can choose $r > 0$ such that $\|f^{(y)} - f\|_2 < \delta$ if $|y| < r$ (see the notes for Lecture 11 for a proof) and we can then choose $\varepsilon_0 > 0$ such that $\int_{|y|>r} \phi_\varepsilon < \delta/(2\|f\|_2)$ when $\varepsilon > \varepsilon_0$. Then we see that $\|f * \phi_\varepsilon - f\|_2 < \delta$ when $\varepsilon < \varepsilon_0$.

Back to the properties of the Fourier transform: (d). Defining $v(x) = \bar{u}(-x)$ and $w = u*v$, we saw last time that $w \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\cap C(\mathbb{R}^n)$ and $w(0) = \|u\|_2^2$, while

$$\hat{w} = (2\pi)^{n/2}|\hat{u}|^2 \qquad \text{so} \qquad \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \hat{w}\,dx \;=\; \|\hat{u}\|_2^2.$$

Hence we want to prove the Fourier inversion formula for w at 0, that is we want to show that

$$w(0) \;=\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{w}\,dx.$$

Set

$$\phi_\varepsilon(x) = \frac{1}{(2\pi)^{n/2}} e^{-\varepsilon|x|^2}.$$

Then

$$\hat{\phi}_\varepsilon(x) \;=\; \frac{1}{(4\pi\varepsilon)} e^{-|x|^2/4\varepsilon}$$

satisfies the conditions of the proposition for being an approximate identity. From (a), we have

$$\int_{\mathbb{R}^n} w\hat{\phi}_\varepsilon \;=\; \int_{\mathbb{R}^n} \hat{w}\phi_\varepsilon.$$

But by the Proposition part (A), the left hand side converges to $w(0)$ as $\varepsilon \to 0$, while since $\phi_\varepsilon \uparrow 1/(2\pi)^{n/2}$ as $\varepsilon \to 0$, by the monotone convergence theorem the right hand side converges to $(1/(2\pi)^{n/2})\int_{\mathbb{R}^n} \hat{w}$ as $\varepsilon \to 0$. This completes the proof of (d).

(e). We will construct a similar argument to the one just given. Assume $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then as before,

$$\int_{\mathbb{R}^n} u\hat{\phi}_\varepsilon \;=\; \int_{\mathbb{R}^n} \hat{u}\phi_\varepsilon.$$

However the left hand side may not converge to $u(0)$ since u may not be continuous at zero, and the right hand side may not converge because $\hat{u}$ may not be in $L^1(\mathbb{R}^n)$. We take a point $z \in \mathbb{R}^n$ and consider the function

$$\psi_\varepsilon(x) = \phi_\varepsilon(x)e^{ix\cdot z}.$$

It is easy to check that the Fourier transform is given by

$$\hat{\psi}_\varepsilon(x) \;=\; \phi_\varepsilon(x-z).$$

Hence by (a), for every $\varepsilon > 0$,

$$(*) \qquad \int_{\mathbb{R}^n} u(x)\hat{\phi}_\varepsilon(x-z)\,dx \;=\; \int_{\mathbb{R}^n} \hat{u}(x)\phi_\varepsilon(x)e^{ix\cdot z}\,dx.$$

Now because ϕ_ε is even (i.e. $\phi_\varepsilon(y) = \phi_\varepsilon(-y)$), the left hand side equals $u * \hat{\phi}_\varepsilon(z)$ which converges to u in $L^2(\mathbb{R}^n)$ as $\varepsilon \to 0$. However, set

$$\rho_\varepsilon(x) \;:=\; \frac{1}{(2\pi)^{n/2}}\phi_\varepsilon(x) \;=\; e^{-|x|^2/\varepsilon}.$$

The right hand side of (*) is

$$(**) \qquad \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{u}(x)\rho_\varepsilon(x)e^{ix\cdot z}\,dx \;=\; \widehat{(\hat{u}\rho_\varepsilon)}(-z).$$

The function $\rho_\varepsilon \uparrow 1$ as $\varepsilon \to 0$, and so by the dominated convergence theorem $\hat{u}\rho_\varepsilon \;\to\; \hat{u}$ in $L^2(\mathbb{R}^n)$ as $\varepsilon \to 0$. Hence by the fact that the Fourier transform is an isometry of L^2, we see that (**) tends to $\check{\hat{u}}$ in $L^2(\mathbb{R}^n)$ as $\varepsilon \to 0$, and the result is proved.