Lecture 13: Fundamental Solution of the Heat Equation

For $u \in L^1(\mathbb{R}^n)$,

$$\hat{u}(x) \;=\; \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x)e^{-ix\cdot y}\,dx, \qquad\qquad \check{u}(x) = \hat{u}(-x).$$

Last time we proved that the Fourier transform extends to an isometry of $L^2(\mathbb{R}^n)$, and we have the Fourier inversion formula $\check{\hat{u}} = u$. We still need to prove (f). If $u, D^\alpha u \in L^2(\mathbb{R}^n)$, then $\widehat{D^\alpha u}(x) \;=\; (ix)^\alpha \hat{u}(x)$.

Proof. First assume that $u \in C_c^\infty(\mathbb{R}^n)$. Then integrating by parts,

$$\begin{aligned}\widehat{D^\alpha u}(y) \;&=\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} D^\alpha u(x)e^{-ix\cdot y}\,dx \;=\; \frac{(-1)^{|\alpha|}}{(2\pi)^n}\int_{\mathbb{R}^n} u(x)D_x^\alpha e^{-ix\cdot y}\,dx \\ &=\; \frac{(iy)^\alpha}{(2\pi)^n}\int_{\mathbb{R}^n} u(x)e^{-ix\cdot y}\,dx \;=\; (iy)^\alpha \hat{u}(y).\end{aligned}$$

Now we want to prove this just assuming that say $u \in L^2(\mathbb{R}^n) \cap C^{|\alpha|}(\mathbb{R}^n)$ and $D^\alpha u \in L^2(\mathbb{R}^n)$. It is not trivial to do an approximation argument. It turns out to be easier to use duality. First notice that again assuming $u \in C_c^\infty(\mathbb{R}^n)$, we have

$$D^\alpha \hat{u} \;=\; \hat{v}, \qquad \text{where} \qquad v(x) = (-ix)^\alpha u(x).$$

Indeed,

$$D^\alpha \hat{u}(y) \;=\; \frac{1}{(2\pi)^n} D_y^\alpha \int_{\mathbb{R}^n} u(x)e^{-ix\cdot y}\,dx \;=\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} u(x)(-ix)^\alpha e^{-ix\cdot y}\,dx.$$

Choose a test function $\phi \in C_c^\infty(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} \hat{u}\phi \;=\; \int_{\mathbb{R}^n} u\hat{\phi}.$$

(We showed this for $u, \phi \in L^1$, but it extends to $u, \phi \in L^2$ by approximating u and ϕ by functions in $L^1 \cap L^2$ and using the boundedness of the Fourier transform on L^2.) Also, defining $\psi(x) = (-ix)^\alpha \phi(x)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \widehat{D^\alpha u}\phi \;&=\; \int_{\mathbb{R}^n} (D^\alpha u)\hat{\phi} \;=\; (-1)^{|\alpha|}\int_{\mathbb{R}^n} uD^\alpha\hat{\phi} \;=\; (-1)^{|\alpha|}\int_{\mathbb{R}^n} u\hat{\psi} \\ &=\; (-1)^{|\alpha|}\int_{\mathbb{R}^n} \hat{u}\psi \;=\; \int_{\mathbb{R}^n} \hat{u}(x)(ix)^\alpha\phi(x)\,dx.\end{aligned}$$

But this is true for every test function ϕ. Hence the equality $\widehat{D^\alpha u} = (ix)^\alpha \hat{u}$ holds in L^2. (It is an exercise to show that if f is locally in L^1 and $\int f\phi = 0$ for every $\phi \in C_c^\infty(\mathbb{R}^n)$, then $f = 0$.)

Now we are ready to solve the heat equation. For a function $u(x,t)$ with $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, write

$$\Delta u \;=\; \sum_j u_{x_j x_j}.$$

Now we try to solve the heat equation

$$\text{(1)} \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x), \end{cases}$$

where $g \in L^2(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Then taking the Fourier transform in x, we have

$$\widehat{\Delta u}(x) \;=\; (\sum_j (ix_j)^2)\hat{u}(x) \;=\; -|x|^2 \hat{u}(x).$$

So

$$\begin{cases} \hat{u}_t + |x|^2 u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ \hat{u}(x,0) = \hat{g}(x). \end{cases}$$

Solving this we get

$$\hat{u}(x,t) \;=\; e^{-|x|^2 t}\hat{g}(x).$$

Set

$$\psi_{(t)}(x) = e^{-|x|^2 t}.$$

Then taking the inverse Fourier transform,

$$u(x,t) \;=\; \check{\psi}_{(t)} * g(x),$$

but

$$\hat{\psi}_{(t)}(x) \;=\; \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/2t}.$$

This shows that (1) is solved by

$$\text{(2)} \qquad u(x,t) \;=\; \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/2t} g(y)\, dy.$$

Theorem. *Suppose that $u \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Then u defined by (2) is in $C6\infty(\mathbb{R}^n \times (0,\infty))$ satisfies*

$$\begin{cases} u_t - \Delta u = 0 & \textit{in } \mathbb{R}^n \times (0,\infty), \\ \lim_{\substack{t \downarrow 0 \\ x \to x_0}} u(x,t) = g(x_0), \end{cases}$$

Proof. Set

$$\Phi(x,t) \;=\; \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/2t} \;=\; 0.$$

We can differentiate under the integral sign in (2) as many times as we like because $\Phi(x,t)$ is smooth and all derivatives are rapidly decaying in x. The fact that

$$(\partial_t - \Delta_x)\Phi(x,t) \;=\; 0$$

can be computed explicitly. The only thing to prove is the boundary behavior. As usual, $\int \Phi(x,t)\,dx = 1$, and

$$\left|\int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\,dy - g(x_0)\right| \;\le\; \left|\int_{\mathbb{R}^n} \Phi(x-y,t)(g(y)-g(x))\,dy - g(x)\right| + |g(x)-g(x_0)|.$$

The first term on the right is bounded by

$$\sup_{|x-y|<r} |g(x)-g(y)| \;+\; 2\|g\|_\infty \int_{|x-y|>r} \Phi(x-y,t)\,dy.$$

Given $\varepsilon > 0$, we first choose $r > 0$ with $r < 1/2$ so that $|g(x)-g(y)| < \varepsilon$ if $|x-y| < r$ for $x, y \in B(x_0, 1)$. Then the terms $|g(x) - g(x_0)|$ and $sup_{|x-y|<r}|g(x) - g(y)|$ are both less than ε. We then choose δ small so that

$$\int_{|x|>r} \Phi(y,t)\,dy \;<\; \varepsilon/(2\|g\|_\infty)$$

when $t < \delta$. This completes the proof.

Fundamental Solution for the Laplacian revisited

If we try to solve $\Delta u = -f$ on $\mathbb{R}^n$, we get

$$|x|^2\hat{u} \;=\; \hat{f},$$

so

$$\hat{u} \;=\; \frac{\hat{f}}{|x|^2}.$$

Hence we expect that writing $\psi(x) = |x|^{-2}$, we have

$$u \;=\; \frac{1}{(2\pi)^{n/2}}\check{\psi} * f.$$

The problem is that ψ is not in L^1 or L^2, but nevertheless we can follow this path. We use the Mellin transform to write,

$$|x|^{-2} \;=\; \int_0^\infty e^{-|x|^2 t}\,dt.$$

Then writing $\phi_t(x) = e^{-|x|^2 t}$, we have

$$
\begin{aligned}
(|x|^2\hat{u})^\vee(y) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_0^\infty \hat{f}(x)e^{-|x|^2t}e^{ix\cdot y}\,dtdx \;=\; \int_0^\infty \check{\phi}_t * f(y)\,dt \\
&= \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}}e^{-|x-y|^2/4t}f(x)\,dxdt \\
&= \left(\int_0^\infty \frac{1}{(4\pi T)^{n/2}}e^{-1/4T}\,dT\right)\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-2}}f(x)\,dx.
\end{aligned}
$$

To get the final expression, we made the change of variables $t \to T$, where $T = t/|x-y|^2$. To show that we get the fundamental solution of Laplace's equation obtained previously, we just have to check the value of the constant:

$$
\int_0^\infty \frac{1}{(4\pi T)^{n/2}}e^{-1/4T}\,dT \;=\; \frac{1}{(n-2)|B^n|}.
$$

We omit this computation.