MATH 231A PARTIAL DIFFERENTIAL EQUATIONS

Lecture 14: Duhamel's Principle

We will use *Duhamel's principle* and the fundamental solution of the homogeneous heat equation

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g, \end{cases}$$

to solve the inhomogeneous heat equation

$$\begin{cases} (\partial_t - \Delta)u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g. \end{cases}$$

We will introduce a more general form of Duhamel's principle than we need, but in a simpler setting. Instead of considering partial differential operators on functions, we start by considering linear operators on vectors in $\mathbb{R}^n$. Suppose the sequence of vectors u_0, u_1, ... in $\mathbb{R}^n$ is determined by the equation

$$u_{j+1} \;=\; A_{j+1}u_j,$$

where A_1, $A_2, \ldots$ is a fixed sequence on linear operators on $\mathbb{R}^N$. It is easy to write down the kth term in this sequence. Indeed,

$$u_k \;=\; A_k A_{k-1} \ldots A_1 u_0.$$

In fact, if $k > j$ then

$$u_k \;=\; A_k A_{k-1} \ldots A_{j+1} u_j.$$

If for $k > j$ we write

$$S(k,j) \;=\; A_k A_{k-1} \ldots A_{j+1},$$

and for convenience we set $S(k,k) = I$. Then

$$u_k \;=\; S(k,j)u_j.$$

Now we seek the solution of the problem

$$u_{j+1} \;=\; A_j u_j \;+\; v_j,$$

where v_0, $v_1, \ldots$ is a given sequence of vectors. We solve this one step at a time as follows.

$$\begin{aligned}
u_1 &= S(1,0)u_0 \;+\; v_1, \\
u_2 &= S(2,0)u_0 \;+\; S(2,1)v_1 \;+\; v_2, \\
u_3 &= S(3,0)u_0 \;+\; S(3,1)v_1 \;+\; S(3,2)v_2 \;+\; v_3, \\
u_4 &= S(4,0)u_0 \;+\; S(4,1)v_1 \;+\; S(4,2)v_2 \;+\; S(4,3)v_3 \;+\; v_4, \\
u_k &= S(k,0)u_0 \;+\; S(k,1)v_1 \;+\; \ldots\ldots\ldots \;+\; S(k,k-1)v_{k-1} \;+\; v_k \\
&= S(k,0)u_0 \;+\; \sum_{j=1}^{k} S(k,j)v_k,
\end{aligned}$$

where, the identity operator. We see that the vector v_j is behaving like additional initial data at time j. This is called *Duhamel's Principle.*

We now we go from difference equations to ordinary differential systems. We remark that that the solution to a linear partial differential system exists for all times. (We did not prove this although it is not hard to prove.)

Theorem. *(Liner ODEs) Suppose that A is a continuous map from $[0,\infty)$ to the space of linear maps on $\mathbb{R}^n$, and suppose that the solution to the ODE*

$$(\partial_t u)(t) \;=\; A(t)u(t), \qquad t\in(0,\infty),$$

is given by

$$u(t) \;=\; S(t,t_0)u(t_0).$$

Then the solution of

$$\begin{cases} \partial_t \;=\; A(t)u(t)+f(t), & t\in(0,\infty),\\ u(0)=g, \end{cases}$$

is given by

$$u(t) \;=\; S(t,0)u(0) \;+\; \int_0^t S(t,s)f(s)\,ds. \tag{1}$$

Proof. With u defined by (1),

$$\begin{aligned}(\partial_t u)(t) \;&=\; A(t)S(t,0)u(t) \;+\; \int_0^t A(t)S(t,s)f(s)\,ds \;+\; S(t,t)f(t).\\ &=\; A(t)\left(S(t,0)u(0) \;+\; \int_0^t S(t,s)f(s)\,ds\right) \;+\; f(t)\\ &=\; A(t)u(t) \;+\; f(t).\end{aligned}$$

As an aside, we can now prove the following Theorem.

Finally, we go from ordinary differential systems to partial differential equations, specifically we solve the non-homogeneous heat equation.

Write $C^{(2,1)}(\mathbb{R}^n\times[0,\infty))$ for the space of real functions $f(x,t)$, such that $\partial_x^\alpha f$ and $\partial_t f$ exist and are continuous for $|\alpha|\le 2$.

Theorem. *If $f\in C^{(2,1)}(\mathbb{R}^n\times[0,\infty))$ has compact support, then $u(x,t)$ defined by*

$$u(x,t) \;=\; \int_0^t \frac{1}{(4\pi(t-s))^{n/2}}\int_{\mathbb{R}^n} e^{-|x-y|^2/4(t-s)}\, f(y,s)\,dyds$$

is in the space $C^{(2,1)}(\mathbb{R}^n\times[0,\infty))$, and solves

$$\begin{cases} (\partial_t-\Delta)u(x,t) \;=\; f(x,t) & x\in\mathbb{R}^n,\ t>0,\\ \lim_{(x,t)\to(x_0,0)} u(x,t)=0, & x_0\in\mathbb{R}^n. \end{cases}$$

Proof. We change variables to obtain

$$u(x,t) \;=\; \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\, f(x-y,t-s)\, dyds$$

The kernel

$$\Phi(y,s) = \frac{1}{(4\pi s)^{n/2}} e^{-|y|^2/4s}$$

is bounded and smooth away from $(y,s) = (0,0)$. Notice that last time when we considered the homogeneous equation, we proved that

$$v(x,s,t) \;=\; \begin{cases} \int_{\mathbb{R}^n} \Phi(y,s)\, f(x-y,t)\, dy & s > 0, \\ f(x,t) \qquad s = 0, \end{cases}$$

is in $C(\mathbb{R}^n \times [0,\infty) \times [0,\infty))$. In the homework, you will see that it is in $C^{(2,0,1)}(\mathbb{R}^n \times [0,\infty) \times [0,\infty))$. Then

$$u(x,t) \;=\; \int_0^t v(x,s,t-s)\, ds$$

is in $C^{(2,1)}(\mathbb{R}^n \times [0,\infty))$, and

$$\begin{aligned} \partial_t u(x,t) \;&=\; v(x,t,0) \;+\; \int_0^t \partial_t v(x,s,t-s)\, ds \\ &=\; \int_{\mathbb{R}^n} \Phi(y,s)\, f(x-y,0)\, dy \;+\; \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\, \partial_t f(x-y,t-s)\, dyds. \end{aligned}$$

Hence

$$\begin{aligned} (\partial_t - \Delta) u(x,t) \;&=\; \int_{\mathbb{R}^n} \Phi(y,s)\, f(x-y,0)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\, (\partial_t - \Delta_x) f(x-y,t-s)\, dyds \\ &=\; (I) \;+\; (II). \end{aligned}$$

Now we write

$$\begin{aligned} (II) \;&=\; \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)\, \partial_t f(x-y,t-s)\, dyds + \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\, \partial_t f(x-y,t-s)\, dyds \\ &=\; (III) \;+\; (IV). \end{aligned}$$

But (III) is bounded by a constant multiple of ε, since $\partial_t f(x-y,t-s)$ is bounded, and $\int_{\mathbb{R}^n} \Phi(y,s)\, dy = 1$. We integrate by parts and use the fact that Φ satisfies the heat equation to compute (IV), and hence we see that $(\partial_t - \Delta) u = f$.

$$\begin{aligned} (IV) \;&=\; \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\, (\partial_t - \Delta_x) f(x-y,t-s)\, dyds \\ &=\; \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\, (-\partial_s - \Delta_y) f(x-y,t-s)\, dyds \\ &=\; \int_\varepsilon^t \int_{\mathbb{R}^n} ((\partial_s - \Delta_y)\Phi)(y,s)\, f(x-y,t-s)\, dyds \\ &\qquad\qquad + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y), t-\varepsilon)\, dy \;-\; \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy \\ &\to f(x,t) \;-\; (I) \text{ as } \varepsilon \to 0. \end{aligned}$$