Lecture 15: Maximum Principle

Last time: If $f \in C^{(2,1)}(\mathbb{R}^n \times [0,\infty))$ has compact support, then $u(x,t)$ defined by

$$u(x,t) \;=\; \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4(t-s)}\, f(y,s)\, dyds$$

is in the space $C^{(2,1)}(\mathbb{R}^n \times [0,\infty))$, and solves

$$\begin{cases} (\partial_t - \Delta)u(x,t) \;=\; f(x,t) & x \in \mathbb{R}^n,\ t > 0, \\ \lim_{(x,t)\to(x_0,0)} u(x,t) = 0, & x_0 \in \mathbb{R}^n. \end{cases}$$

It remains to check the boundary behavior, but this is clear since

$$\frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4(t-s)}\, dy \;=\; 1,$$

so

$$\left| \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4(t-s)}\, f(y,s)\, dy \right| \;\le\; \sup_x |f(x,s)|,$$

and

$$|u(x,t)| \;\le\; t \sup_{0\le s\le t} \sup_x |f(x,s)| \;\to\; 0,$$

as $t \to 0$.

Notation. Let U be a bounded open set in $\mathbb{R}^n$. Set

$$U_T \;=\; \{(x,t) : x \in U,\ 0 < t \le T\}.$$

Set

$$\Gamma_t \;=\; \bar{U}_T \setminus U_T \;=\; (\partial U \times [0,T]) \;\cup\; (U \times \{0\}).$$

Maximum Principle. *Suppose $u \in C^{(2,1)}(U_T) \cap C(\bar{U}_T)$ solves*

$$(\partial_t - \Delta)u \;=\; 0, \qquad \textit{in } U_T.$$

Then u attains its maximum on Γ_T.

Proof. Define $u_\varepsilon(x,t) = u(x,t) - \varepsilon t$. Since $\bar{U}_T$ is compact and u_ε is continuous, u_ε attains its maximum at some point $(x_\varepsilon, t_\varepsilon) \in \bar{U}_T$. Suppose that $(x_\varepsilon, t_\varepsilon) \in U_T$. Then

(*) $$\Delta u_\varepsilon(x_\varepsilon, t_\varepsilon) \le 0, \qquad \partial_t u_\varepsilon(x_\varepsilon, t_\varepsilon) \ge 0.$$

However,

$$\partial_t u_\varepsilon \;=\; \partial_t(u - \varepsilon t) \;=\; \Delta u - \varepsilon,$$

so

$$\partial_t u \ < \ \Delta u,$$

and (*) is a contradiction. Hence $(x_\varepsilon, t_\varepsilon) \in \Gamma_\varepsilon$. Now on $\bar{U}_T$,

$$u_\varepsilon(y,s) \ \leq \ u_\varepsilon(x_\varepsilon, t_\varepsilon),$$

so

$$u(y,s) \ - \ \varepsilon s \ \leq \ u(x_\varepsilon, t_\varepsilon) \ - \ \varepsilon t_\varepsilon$$

and

$$u(y,s) \ \leq \ u(x_\varepsilon, t_\varepsilon) \ + \ \varepsilon T.$$

But since Γ_ε is compact, there exists a subsequence $\varepsilon_j \to 0$ as $j \to \infty$, such that

$$(x_{\varepsilon_j}, t_{\varepsilon_j}) \ \to \ (x,t) \in \Gamma_T \qquad \text{as } j \to \infty.$$

But then for $(y,s) \in \bar{U}_T$,

$$u(y,s) \ \leq \ \lim_{j\to\infty} \ u(x_\varepsilon, t_\varepsilon) + \varepsilon T \ = \ u(x,t).$$

Uniqueness Theorem. *There is at most one solution $u \in C^{(2,1)}(U_T) \cap C(\bar{U}_T)$ to the equation*

$$\begin{cases} (\partial_t - \Delta)u = f & \text{in } U_T, \\ u = g & \text{on } \Gamma_T. \end{cases}$$

Proof. If $u, \tilde{u}$ are solutions, then $v = u - \tilde{u}$ satisfies

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } U_T, \\ u = 0 & \text{on } \Gamma_T. \end{cases}$$

By the maximum principle $v \leq 0$ on $\bar{U}_T$, but applying the maximum principle to $-v$ we get $-v \leq 0$ on $\bar{U}_T$, so $u = 0$ on $\bar{U}_T$.

Uniqueness Theorem. *There is at most one solution $u \in C^{(2,1)}(\mathbb{R}^n) \times (0,T]) \cap C(\mathbb{R}^n \times [0,t])$ to the equation*

$$\begin{cases} (\partial_t - \Delta)u = f & \text{in } \mathbb{R}^n \times (0,T], \\ u(x,0) = g, \end{cases}$$

such that there exists C and A with $u(x,t) \leq Ce^{A|x|^2}$ on $\mathbb{R}^n \times [0,T]$.

Proof. If u and $\tilde{u}$ are two solutions, consider $w = u - \tilde{u}$. Then w satisfies

$$\begin{cases} (\partial_t - \Delta)w = 0 & \text{in } \mathbb{R}^n \times (0,T], \\ u(x,0) = 0, \end{cases}$$

and a growth estimate of the form $|w(x,t)| \le Ce^{A|x|^2}$. Assume to start with that $4AT < 1$, and choose ε such that $4A(T+\varepsilon) < 1$, and for $\eta > 0$, consider the function

$$v(x,t) \;=\; w(x,t) \;-\; \frac{\eta}{(T+\varepsilon-t)^{n/2}} e^{|x|^2/(4(T+\varepsilon-t))}.$$

This satisfies the heat equation and is negative when $t = 0$. Moreover,

$$|v(x,t)| \;\le\; Ce^{A|x|^2} - \frac{\eta}{(T+\varepsilon)^{n/2}} e^{|x|^2/(4(T+\varepsilon))} \;=\; e^{A|x|^2}\left(C - \frac{\eta}{(T+\varepsilon)^{n/2}} e^{|x|^2(1/(4(T+\varepsilon))-A)}\right)$$

and this is negative for $|x|$ sufficiently large, so for R large, setting $U = B(0,R)$, we have v negative on Γ_T, so by the maximum principle, v is negative on U_T. Taking $R \to \infty$, we get that v is non-positive on $\mathbb{R}^n \times [0,T]$, and so

$$w(x,t) \;\le\; \frac{\eta}{(T+\varepsilon-t)^{n/2}} e^{|x|^2/(4(T+\varepsilon-t))}.$$

However, this is true for all $\eta > 0$, so taking $\eta \to 0$ we get that w is non-positive. The same is true for $-w$, so $w = 0$.

In the case that $4AT > 1$, split $[0,T]$ into intervals $[0,T_1]$, $[T_1,T_2], \ldots, [T_{m-1},T_m]$ with $4A(T_j - T_{j-1}) < 1$ and apply the result to each interval.