MATH 231A PARTIAL DIFFERENTIAL EQUATIONS

Lecture 16: Energy Methods

Last time, we proved uniqueness to the equation

$$\begin{cases} (\partial_t - \Delta)u = f & \text{in } \mathbb{R}^n \times (0,T], \\ u(x,0) = g, \end{cases}$$

under the assumption that $u \in C^{(2,1)}(\mathbb{R}^n) \times (0,T]) \cap C(\mathbb{R}^n \times [0,t])$, and there exists C and A with $u(x,t) \leq Ce^{A|x|^2}$.

A closely related result is the **maximum principle**: For a u satisfying the above regularity and growth conditions which solves the heat equation

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } \mathbb{R}^n \times (0,T], \\ u(x,0) = g, \end{cases}$$

we have $u \leq \max g$ on $\mathbb{R}^n \times [0,T]$. To sketch the proof, first assume $4AT < 1$, or cut $[0,T]$ into interval $[0,T_1], \ldots, [T_{m-1}, T_m]$ so that $4A(T_j - T_{j-1}) < 1$. Assuming $4AT < 1$, consider

$$v(x,t) \;=\; u(x,t) \;-\; \frac{\eta}{(T+\varepsilon-t)^{n/2}} e^{|x|^2/4(T+\varepsilon-t)},$$

which is less than u and still satisfies the heat equation. We find that $v(x,t) \to -\infty$ as $|x| = R \to \infty$, so taking R large enough so that $v(x,t) \leq \max g$ when $|x| = R$, by the maximum principle $v(x,t) \leq \sup g$ on $B(0,R) \times [0,T]$. Letting $R \to \infty$ we get $v \leq \sup g$ and then taking $\eta \to 0$, we get $u \leq \sup g$. The uniqueness theorem can be deduced from this.

We need the growth condition on u to get uniqueness. Indeed, we will now give some non-trivial solutions to

$$\begin{cases} (\partial_t - \partial_x^2)u = 0 & \text{in } \mathbb{R} \times (0,T], \\ u(x,0) = 0, \end{cases}$$

We will prescribe on the t-axis,

$$u(0,t) = g(t), \qquad u_x(0,t) = 0,$$

for certain functions g. We try to express

$$u(x,t) \;=\; \sum_{j=0}^{\infty} g_j(t) x^j,$$

where $g_0 = g$ and $g_1 = 0$. The heat equation gives

$$\sum g_j'(t) j x^j \;-\; \sum g_j(x) j(j-1) x^{j-2} \;=\; 0,$$

so

$$g_{j+2} \;=\; \frac{g_j'}{(j+2)(j+1)}.$$

This leads to

$$u(x,t) \;=\; \sum_{j=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Set for example, for $\alpha > 1$,

$$g(t) \;=\; \begin{cases} e^{-1/t^{\alpha}}, & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Then by bounded the size of the derivatives of g, it can be shown that the series for $u(x,t)$ converges and is in $C^{\infty}(\mathbb{R} \times \mathbb{R})$.

Uniqueness Theorem with energy method. *Suppose that $u \subset \mathbb{R}^n$ is open and bounded. There exists at most one solution $u \in C^(2,1)(\bar{U}_T)$ to the equation*

$$\begin{cases} (\partial_t - \Delta)u = f & \textit{in } U_T, \\ u = g & \textit{on } \Gamma_T. \end{cases}$$

Proof. If u and $\bar{u}$ are both solutions, then $w = u - \bar{u}$ solves

$$\begin{cases} (\partial_t - \Delta)w = 0 & \text{in } U_T, \\ w = 0 & \text{on } \Gamma_T. \end{cases}$$

Consider

$$e(t) \;=\; \int_U w^2(x,t)\,dx \;\geq\; 0.$$

Then $e \in C^1[0,T]$, and

$$\dot{e}(t) \;=\; 2\int_U w\dot{w}\,dx \;=\; 2\int_U w\Delta w\,dx \;=\; -2\int_U |\nabla w|^2\,dx \;\leq\; 0.$$

But $e(0) = 0$, hence $e(t) = 0$ for all t, and $w = 0$ on $\bar{U}_T$.

For an open bounded set $U \subset \mathbb{R}^n$, set Λ_T to be the sides plus the *top* of U_T, so

$$\Lambda_T \;=\; (\partial U \times [0,T]) \;\cup\; (U \times \{T\}).$$

Backwards uniqueness. . *There exists at most one solution $u \in C^2(\bar{U}_T)$ to the equation*

$$\begin{cases} (\partial_t - \Delta)u = f & \textit{in } U_T, \\ u = g & \textit{on } \Lambda_T. \end{cases}$$

Proof. Suppose u and $\tilde{u}$ are two solutions and set $w = u - \tilde{u}$ which satisfies

$$\begin{cases} (\partial_t - \Delta)w = 0 & \text{in } U_T, \\ w = 0 & \text{on } \Lambda_T. \end{cases}$$

Set

$$e(t) \;=\; \int_U w^2(x,t)\,dx.$$

Then as before,

$$\dot{e}(t) \;=\; -2\int_U \nabla w \cdot \nabla w\,dx.$$

But then

$$\ddot{e}(t) \;=\; -4\int_U \dot{\nabla} w \cdot \nabla w\,dx \;=\; 4\int_U \dot{w}\Delta w\,dx \;=\; 4\int_U (\Delta w)^2\,dx \;\geq 0.$$

Now by Hölder's inequality,

$$\int |\nabla w|^2 \;\leq\; \int w\Delta w \;\leq\; \left(\int w^2\right)^{1/2} \left(\int (\Delta w)^2\right)^{1/2},$$

so

$$(\dot{e}(t))^2 \;=\; 4\left(\int |\nabla w|^2\,dx\right)^2 \;\leq\; 4\left(\int w^2\,dx\right)^2 \left(\int (\Delta w)^2\,dx\right) \;=\; \ddot{e}(t)e(t).$$