Math 231A Partial Differential Equations

Lecture 17: Smoothness of Solutions

Last time we were showing backwards uniqueness:

Backwards uniqueness. . *If U is open and bounded in $\mathbb{R}^n$, then there exists at most one solution $u \in C^2(\bar{U}_T)$ to the equation*

$$\begin{cases} (\partial_t - \Delta)u = f & \text{in } U_T, \\ u = g & \text{on } \Lambda_T. \end{cases}$$

So far we considered the difference $w = u - \tilde{u}$ of two solutions and defined

$$e(t) \;=\; \int_U w^2(x,t)\,dx.$$

We showed using Cauchy-Schwarz that

$$(\dot{e}(t))^2 \;\le\; \ddot{e}(t)e(t).$$

Now we have $e(T) = 0$. Suppose that there exists $t \in [0,T]$ with $e(t) > 0$. Then there exists an interval $[t_1, t_2]$ with $e(t) > 0$ for $t_1 \le t < t_2$ and $e(t_2) = 0$. Indeed, the set $J = \{t : e(t) > 0\}$ is open, and we can take some maximal open interval (t_1, t_2) contained in J. Then $e(t_2) = 0$ since this interval is maximal. Now define

$$f(t) \;=\; \log e(t), \qquad\qquad t_1 < t < t_2.$$

Then

$$\ddot{f} \;=\; \left(\frac{\dot{e}}{e}\right)^{\cdot} \;=\; \frac{\ddot{e}}{e} \;-\; \frac{(\dot{e})^2}{e^2} \;=\; \frac{\ddot{e}e - (\dot{e})^2}{e} \;\ge\; 0,$$

so f is convex on (t_1, t_2). Then for $t_1 < s < t < t_2$, and $0 < \tau < 1$,

$$f(\tau s + (1-\tau)t) \;\le\; \tau f(s) \;+\; (1-\tau)f(t).$$

But then

$$e(\tau s + (1 - \tau t)) \;\le\; e(s)^{\tau}\; e(t)^{1-\tau}.$$

But letting $t \to t_2$, this gives

$$e(\tau s + (1 - \tau t_2)) \;\le\; 0.$$

Hence e vanishes on (t_1, t_2) which is a contradiction.

Theorem. *If $u \in C^{(2,1)}(U_T)$ solves the heat equation on U_T. Then $u \in C^\infty(U_T)$.*

Proof.
First assume that $u \in C^{(4,2)}(\mathbb{R}^n)$. We will remove this condition later.

To show that u is smooth on a neighborhood of a point $(x_0,t_0) \in U_T$, start by choosing $r > 0$ so that for the ball $B_{2r}(x_0,t_0)$ in $\mathbb{R}^{n+1}$ does not intersect Γ_T. Set $B = B_{2r}(x_0,t_0) \cap U_T$, and choose a smooth cut-off function $\chi \in C_c^\infty(\mathbb{R}^{n+1})$ such that $\chi = 1$ on B, and $\chi = 0$ outside $B_{2r}(x_0,t_0)$. Then

$$f(x,t) \ := \ (\partial_t - \Delta)(\chi u)(x,t) \ \in \ C^{(2,1)}(\mathbb{R}^n \times [0,T]),$$

and $\chi u = 0$ on $\mathbb{R}^n$, so by the uniqueness Theorem,

$$(*) \qquad \psi(x,t)u(x,t) \ = \ \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dyds.$$

Indeed, if $v(x,t)$ is the right hand side of (*), then by previous results, $v \in C^{(2,1)}(\mathbb{R}^n \times [0,\infty))$, and v satisfies

$$\begin{cases} (\partial_t - \Delta) = f, & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = 0. \end{cases}$$

But $\tilde{v}(x,t) = \psi(x,t)u(x,t)$ also satisfies these equations, and $\tilde{v}$ is compactly supported so is bounded. Hence by the uniqueness theorem on $\mathbb{R}^n$ we have $\tilde{v} = v$.

Hence when $(x,t) \in B$ we have

$$(*) \qquad u(x,t) \ = \ \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dyds.$$

Now the kernel $\Phi(x-y,t-s)$ is smooth on $\mathbb{R}^n \times [0,t]$ away from the single point $(y,s) = (x,t)$. However, we see that $f(y,s)$ vanishes on B, indeed,

$$\begin{aligned} f \ = \ (\partial_t - \Delta)(\chi u) \ &= \ \chi(\partial_t - \Delta)u \ + \ u(\partial_t - \Delta)\chi \ - \ \nabla\chi \cdot \nabla u \\ &= \ u(\partial_t - \Delta)\chi \ - \ \nabla\chi \cdot \nabla u. \end{aligned}$$

But any derivative of χ vanishes on B. Hence for $(x,t) \in B$, the integrand in (*) is smooth in (x,t) and supported on the fixed compact set $\bar{U}_T \setminus B$. From this we can see that u is smooth on B. This was however under the assumption that $u \in C^{(4,2)}(U_T)$. If this is not the case, we need to be more careful. Still assuming $u \in C^{(4,2)}(U_T)$, integrating by parts we have

$$\begin{aligned} u(x,t) \ &= \ \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) \left(\left((\partial_t - \Delta)\chi(y,s) \right) u(u,s) \ - \ \nabla\chi \cdot \nabla u \right) \, dyds \\ &= \ \int_0^t \int_{\mathbb{R}^n} \left(\Phi(x-y,t-s)(\partial_t - \Delta)\chi(y,s) \ + \ \nabla_y \Phi(x-y,t-s) \cdot \nabla\chi(y,s) \right) \, u(y,s)\, dyds. \end{aligned}$$

This formula holds assuming $u \in C^{(4,2)}(U_T)$. Now if $u \in C^{(2,1)}(U_T)$, form the convolution on $\mathbb{R}^{n+1}$ given by $u_\varepsilon = \phi_\varepsilon * u$, where now ϕ_ε is defined by

$$\phi_\varepsilon(x,t) \;=\; \frac{1}{\varepsilon^{n+1}}\phi(x/\varepsilon, t/\varepsilon),$$

for some non-negative function $\phi \in C_c^\infty(\mathbb{R}^{n+1})$ with $\int_{\mathbb{R}^{n+1}} \phi = 1$. We see that u_ε is smooth (on a slightly smaller set than U_T) and satisfies the heat equation, so

$$u_\varepsilon(x,t) \;=\; \int_0^t \int_{\mathbb{R}^n} \left(\Phi(x-y,t-s)(\partial_t - \Delta)\chi(y,s) \;+\; \nabla_y \Phi(x-y,t-s)\cdot\nabla\chi(y,s)\right) u_\varepsilon(y,s)\, dyds.$$

But taking $\varepsilon \to 0$, in the limit we get that this equation holds for u, from which we see that u is smooth.