MATH 231A PARTIAL DIFFERENTIAL EQUATIONS

Lecture 18: The Wave Equation

In the first lecture, we derived the equation of the vibrating string $u_{tt} - u_{xx} = 0$. We will not solve this equation, and the generalization $u_{tt} - \Delta u = 0$ to n space dimensions, which describes the propagation of waves in space. We are now going to solve these equations.

$n = 1$.

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g, \ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

To solve this, we factor

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u \;=\; 0.$$

Setting

$$v \;=\; \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u,$$

so

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) v \;=\; 0.$$

This is the homogeneous transport equation and we recall the solution is given by noticing

$$\frac{d}{ds} v(x + s, t + s) = 0,$$

so

$$v(x, t) \;=\; v(x - t, 0) \;=\; a(x - t),$$

where

$$a(x) = v(x, 0) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\bigg|_{(x,0)} u(x, t) \;=\; h(x) - g'(x).$$

Now we solve this equation:

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x, t) \;=\; a(x - t).$$

We solve this transport equation by noticing

$$\frac{d}{ds} u(x - s, t + s) \;=\; a(x - t - 2s),$$

so

$$\begin{aligned} u(x, t) \;&=\; u(x + t, 0) \;+\; \int_{s=-t}^{0} a(x - t - 2s)\, ds \;=\; u(x + t, 0) \;+\; \frac{1}{2}\int_{y=x-t}^{x+t} a(y)\, dy \\ &=\; g(x + t) \;+\; \frac{1}{2}\int_{y=x-t}^{x+t} h(y) - g'(y)\, dy \\ &=\; \frac{1}{2}(g(x + t) + g(x - t)) \;+\; \frac{1}{2}\int_{y=x-t}^{x+t} h(y)\, dy. \end{aligned}$$

This is known as d'Alembert's formula.

Theorem. . *If $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$, then*

$$u(x,t) \;=\; \frac{1}{2}\int_{y=x-t}^{x+t} h(y) - g'(y)\,dy\frac{1}{2}(g(x+t)+g(x-t)) \;+\; \frac{1}{2}\int_{y=x-t}^{x+t} h(y)\,dy$$

is in $C^2(\mathbb{R}\times[0,\infty)$ and solves

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g,\;\; u_t = h & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

This is easy to check. Notice that $u(x,t) = F(x+t) + G(x-t)$, and that any function of this form is a solution. This is due to the factorization.

We also see that if the initial data g is in $C^k(\mathbb{R})$ and $h = 0$, then $u \in C^k(\mathbb{R}\times(0,\infty)$, but we don't get any improvement in the smoothness. This contrasts with the situation for the heat equation, where for bounded continuous initial data on $\mathbb{R}^n$, we got $u \in C^\infty(\mathbb{R}^n \times (0,\infty)$.

Notice also what happens to a small blip in an infinite string which is released from rest at time $t = 0$. The blip splits into two equal pieces. One piece travels to the right at unit speed, and the other one travels to the left at unit speed. If, on the other hand the string starts at rest and is hit on a neighborhood of the point 0 with a certain velocity, then this impact causes the string to be displaced on the interval between $-t$ and $+t$. In both cases, we see that the effect of the initial disturbance travels at unit speed. This is called finite speed of propagation. For the heat equation we have infinite speed of propagation.

String is fixed at one end: This is similar to the problem we studied using Fourier series, where the string is fixed at two endpoints. We want to solve

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = g, \quad u_t = h & \text{on } \mathbb{R}_+ \times \{t=0\} \\ u = 0 & \text{on } \{x=0\} \times (0,\infty), \end{cases}$$

where $f(0) = g(0) = 0$. We reflect the problem by setting

$$\tilde{u}(x,t) \;=\; \begin{cases} u(x,t) & x \geq 0 \\ -u(-x,t) & x < 0, \end{cases}$$

$$\tilde{g}(x) \;=\; \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x < 0, \end{cases}$$

$$\tilde{h}(x) \;=\; \begin{cases} h(x) & x \geq 0 \\ -h(-x) & x < 0, \end{cases}$$

Then $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ \tilde{u} = \tilde{g}, \quad \tilde{u}_t = \tilde{h} & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

so

$$\tilde{u}(x,t) \;=\; \frac{1}{2}(\tilde{g}(x+t) + \tilde{g}(x-t)) \;+\; \frac{1}{2}\int_{y=x-t}^{x+t} \tilde{h}(y)\,dy.$$

We then have

$$u(x,t) \;=\; \begin{cases} \frac{1}{2}(\tilde{g}(x+t) + \tilde{g}(x-t)) \;+\; \frac{1}{2}\int_{y=x-t}^{x+t} \tilde{h}(y)\,dy & t \le x < \infty, \\ \frac{1}{2}(\tilde{g}(x+t) - \tilde{g}(t-x)) \;+\; \frac{1}{2}\int_{y=t-x}^{x+t} \tilde{h}(y)\,dy & 0 \le x \le t. \end{cases}$$

Again look at a small blip in the string which is released from rest. It splits into two equal parts. One part travels towards infinity with unit speed. The other part travels towards zero with unit speed. When it reaches zero, it is reflected and starts traveling towards infinity with unit speed.