Lecture 19: Spherical means

Spherical Means. We now want to solve

$$(*)\qquad \begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g, \qquad u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

The idea is for n odd to reduce this to the one dimensional case, and for n even to work in $n+1$ space dimensions with data which is constant in the last variable. The first step in solving the problem is to suppose that we have a C^2 solution $u(x,t)$ to (*), and form the spherical means. Define

$$U(x,r,t) \;=\; \fint_{\partial B(x,r)} u(y,t)\, dS(y).$$

Then similarly

$$G(x,r) \;=\; \fint_{\partial B(x,r)} g(y)\, dS(y),$$

and

$$H(x,r) \;=\; \fint_{\partial B(x,r)} h(y)\, dS(y).$$

Euler-Poisson-Darboux. *For fixed x, if $u \in C^2(\mathbb{R}^n \times [0,\infty))$, then $U \in C^2(\mathbb{R}^n \times [0,\infty) \times [0,\infty))$, and*

$$(*)\qquad \begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \text{in } \mathbb{R}_+ \times (0,\infty), \\ U = G, \quad U_t = H & \text{on } \mathbb{R}_+ \times \{0\}. \end{cases}$$

The way we think of (*) as saying that from a solution u to the wave equation, we can obtain a spherically symmetric solution by averaging u over spheres. Indeed, with x fixed, for $y \in \mathbb{R}^n$, define $V(y,t) = U(x, |x-y|, t)$. Then $V(y,t)$ is the average of u over the sphere center x which passes through the point y. Then (*) says that V satisfies the wave equation. Indeed since V depends only on $r = |x-y|$,

$$\Delta_y V \;=\; \left(\partial_r^2 \;+\; \frac{n-1}{r}\partial_r\right) U.$$

To prove (*), we have

$$U_r(x,r,t) \;=\; \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t)\, dy \;=\; \frac{r^{1-n}}{n|B(0,1)|} \int_{B(x,r)} \Delta u(y,t)\, dy.$$

Now

$$\partial_r \int_{B(x,r)} \Delta u(y,t)\, dy \;=\; \partial_r \int_0^r \int_{\partial B(x,s)} \Delta u(y,t)\, dS(y)\, ds \;=\; \int_{\partial B(x,r)} \Delta u(y,t)\, dS(y).$$

So using $n|B(0,1)| = |\partial B(0,1)|$, and $\Delta u = u_{tt}$.

$$(r^{n-1}U_r)_r \;=\; \frac{1}{n|B(0,1)|}\int_{\partial B(x,r)} \Delta u(y,t)\,dS(y) \;=\; r^{n-1} ⨍_{\partial B(x,r)} u_{tt}(y,t)\,dS(y) \;=\; r^{n-1}U_{tt}.$$

Solution for $n = 3$. We now wish to obtain a solution of the one dimensional wave equation from the radial solution U of the 3 dimensional wave equation. The key is in noticing that

$$U_{rr} \;+\; \frac{2}{r}U_r \;=\; \frac{1}{r^2}(r^2U_r)_r \;=\; \frac{1}{r}(rU)_{rr}.$$

Thus $\tilde{U} = rU$ with U defined above satisfies

$$\tilde{U}_{rr} \;=\; (rU)_{rr} \;=\; (rU)_{tt} \;=\; \tilde{U}_{tt},$$

so $\tilde{U}$ satisfies the one dimensional wave equation. What are the boundary data?

$$\begin{cases} rU = rG, & \partial_t(rU) = rH \qquad t = 0, \\ rU = 0, & \qquad\qquad\qquad r = 0. \end{cases}$$

Set $\tilde{G}(r) = rG(x,r)$ and $\tilde{H}(r) = rH(x,r)$. Hence for $0 \le r \le t$ we have

$$\tilde{U}(x,r,t) \;=\; \frac{1}{2}\left(\tilde{G}(r+t) - \tilde{G}(t-r)\right) \;+\; \frac{1}{2}\int_{t-r}^{r+t} \tilde{H}(s)\,ds.$$

However,

$$u(x,t) \;=\; \lim_{r\to 0} U(x,r,t) \;=\; \lim_{r\to 0}\frac{\tilde{U}(x,r,t)}{r} \;=\; \tilde{G}'(t) \;+\; \tilde{H}(t) \;=\; \frac{\partial}{\partial t}\left(t ⨍_{\partial B(x,t)} g(y)\,dS(y)\right) \;+\; t ⨍_{\partial B(x,t)} h(y)\,dS(y).$$

Evaluating the derivative, we have

$$\frac{\partial}{\partial t} ⨍_{\partial B(x,t)} g(y)\,dS(y) \;=\; \frac{\partial}{\partial t} ⨍_{\partial B(0,1)} g(x+ty)\,dS(y) \;=\; ⨍_{\partial B(0,1)} \nabla g(x+ty)\cdot y\,dS(y) \;=\; ⨍_{\partial B(x,t)} \nabla g(y)\cdot(y-x)\,dS(y)$$

Hence

$$u(x,t) \;=\; ⨍_{\partial B(x,t)} (g(y) \;+\; \nabla g(y)\cdot(y-x) \;+\; th(y))\,dS(y).$$