

Lecture 2: Classifying Equations.

Before going on, we finish our discussion of Fourier Sine series and the vibrating string.

Suppose $f : [0, \pi] \rightarrow \mathbb{R}$. When is

$$(1) \quad f(x) = \sum_{j=1}^{\infty} \hat{f}_j \sin jx,$$

where

$$\hat{f}_j = \frac{2}{\pi} \int_0^{\pi} f(x) \sin jx \, dx?$$

Define

$$\|f\|_2 = \sqrt{\frac{2}{\pi} \int_0^{\pi} |f|^2 \, dx},$$

and define

$$L^2[0, \pi] = \{f : [0, \pi] \rightarrow \mathbb{R} : \|f\|_2 < \infty\}.$$

(We assume the functions f are measurable, or we take continuous functions and then take the completion of the resulting space - refer to real analysis texts.) Then (1) holds with the series converging in the norm $\| \cdot \|_2$. The Plancherel theorem states that

$$\|f\|_2^2 = \sum_{j=1}^{\infty} |\hat{f}_j|^2.$$

We are going to prove uniqueness of the solution to the vibrating string problem, just using the Plancherel theorem.

Remark. While it is difficult to describe the Fourier Sine coefficients of a continuous function, we see that it is easy to describe the coefficients of a square integrable function. For problems involving Fourier expansions one often prefers not to work with C^k functions (k times continuously differentiable) but instead one works with functions whose derivatives up to the k th order are square integrable. The space of such functions is called a *Sobolev space*. We will see them in the Winter. We will also need to make sense of the concept of a function having a square integrable derivative. Such a derivative is an example of a *distributional derivative* and it is defined using *duality*. Roughly speaking, $h(x)$ is the derivative of $f(x)$ if

$$(2) \quad \int hg = - \int fg'$$

for all smooth functions g of compact support. The point here is that if f is C^1 , then $h = f'$ will satisfy (2) by integration by parts. However, many times a function h can be found to satisfy (2) even when f is not C^1 .

Now suppose that $u(x, t)$ is twice continuously differentiable and solves

$$\begin{cases} u_{xx} = u_{tt} & x \in [0, \pi], \quad t \geq 0 \\ u(x, 0) = g(x) & x \in [0, \pi] \\ u_t(x, 0) = h(x) & x \in [0, \pi] \\ u(0, t) = u(\pi, t) = 0 & t \geq 0. \end{cases}$$

By the Plancherel theorem, for each fixed t we have

$$u(x, t) = \sum_{j=1}^{\infty} \sin jx,$$

where

$$(3) \quad u_j(t) = \frac{2}{\pi} \int_0^{\pi} u(x, t) \sin jx \, dx.$$

Then integrating by parts twice,

$$\begin{aligned} \ddot{u}_j(t) &= \frac{2}{\pi} \int_0^{\pi} u_{tt}(x, t) \sin jx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} u_{xx}(x, t) \sin jx \, dx = -j^2 \int_0^{\pi} u(x, t) \sin jt \, dx = -j^2 u_j(t). \end{aligned}$$

Hence solving this ODE,

$$(4) \quad u_j(t) = a_j \cos jt + b_j \sin jt.$$

Now $u(x, 0) = b(x)$ so $u_j(0) = \hat{b}_j$ and setting $t = 0$ in (4) we see that $a_j = \hat{g}_j$. Similarly, differentiating (3) with respect to t at $t = 0$ gives $\dot{u}_j(0) = \hat{h}_j$ and from (4) we get $\hat{h}_j = jb_j$. Thus the solution is uniquely given by

$$u(x, t) = \sum_{j=1}^{\infty} \left(\hat{g}_j \cos jt + \frac{\hat{h}_j}{j} \sin jt \right) \sin jx.$$

This is physically significant because we have succeeded in writing a general vibration of the string as a superposition of vibrations of different frequencies. In real life the values of these frequencies agree well with experiment, although the vibrations quickly die down due to damping forces that we did not include in our equation.

Notation. A *multi-index* is an n -tuple $\alpha = (\alpha_1, \dots, \alpha_n)$, where α_j are natural numbers. The *order* of α is

$$|\alpha| = \alpha_1 + \dots + \alpha_n.$$

Let $U \subset \mathbb{R}^n$. For $u : U \rightarrow \mathbb{R}$, define $D^\alpha u : U \rightarrow \mathbb{R}$ by

$$D^\alpha u := \partial_x^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u.$$

For $k = 0, 1, 2, \dots$, write

$$D^k u = (D^\alpha u)_{|\alpha|=k}.$$

Exercise. Let $c(n, k)$ denote the number of multiindices α with $|\alpha| = k$. Compute $c(n, k)$.

A *partial differential equation (PDE) of order k* is an equation of the form

$$F(D^k u, D^{k-1} u, \dots, u, x) = 0,$$

where $F : \mathbb{R}^{c(n,k)} \times \mathbb{R}^{c(n,k-1)} \times \dots \times \mathbb{R} \times U \rightarrow \mathbb{R}$ is a function. When F and u take values in \mathbb{R}^m instead of \mathbb{R} , then F is called a *partial differential system*. We will not consider systems in this course.

Classification. PDE are classified as linear, semilinear, quasilinear and fully non-linear. Some PDE can be classified as elliptic, parabolic or hyperbolic.

Linear:

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x) = f(x).$$

Example The equation

$$T(x, t) u_{xx} = m(x, t) u_{tt}$$

is linear. The equation

$$u_{xx} = u_{tt}$$

is linear with constant coefficients.

Semilinear:

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(D^{k-1} u, \dots, u, x) = 0.$$

Quasilinear:

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1} u, \dots, u, x) D^\alpha u(x) + F(D^{k-1} u, \dots, u, x) = 0.$$

Example The equation

$$\left(\frac{u_x}{\sqrt{1 + u_x^2}} \right)_x = u_{tt}$$

is quasilinear because the left hand side is

$$\frac{u_{xx}}{(1 + u_x^2)^{3/2}}.$$

Fully non-linear: None of the above. Depends non-linearly on highest order terms.

Example. Let us suppose that the mass and tension in the vibrating string problem depend only on u_x . Then the vibrating string equation

$$\left(T(x, u_x) \frac{u_x(x, t)}{\sqrt{1 + u_x^2(x, t)}} \right)_x = m(x, u_x) u_{tt}(x, t)$$

could be fully non-linear depending on T and m .

Example. The Monge-Ampère equation

$$\det(D^2u) = f$$

is fully non-linear.

Elliptic, parabolic, hyperbolic.

The operator L with $Lu = u_{tt} + u_{xx}$ is an example of an *elliptic* operator. $Lu = 0$ is the 2 dimensional *Laplace equation*.

The operator $Lu = u_{tt} - u_{xx}$ is an example of a *hyperbolic* operator. $Lu = 0$ is the one space dimensional *wave equation*.

The operator $Lu = u_t - u_{xx}$ is an example of a *parabolic* operator. $Lu = 0$ is the one space dimensional *heat equation*.