Math 231A Partial Differential Equations

Lecture 20: Method of Descent

Some remarks about our solutions to the wave equation:

$$(*)\qquad \begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x). \end{cases}$$

Definition. (a). If $U \subset \mathbb{R}^n$, the *domain of influence* of $U \times \{t_0\}$ is the set of points $(x,t) \in \mathbb{R}^n \times [0,\infty)$ such $u(x,t) \neq 0$ for some solution u to the wave equation which vanishes on $(R^n \setminus U) \times \{t_0\}$.
(b). If $(x_0, t_0) \in \mathbb{T}^n \times [0,\infty)$, the *domain of influence* of (x_0,t_0) is intersection of the domains of influence of the sets $B(x_0,\varepsilon) \times \{t_0\}$ for $\varepsilon > 0$.
(c). If $(x_0,t_0) \subset \mathbb{R}^n \times [0,\infty)$, the *domain of dependence* of (x_0,t_0) is the set of points (x,t) whose domain of influence includes (x_0,t_0).

When $n = 1$, the solution is

$$u(x,t) \;=\; \frac{1}{2}(g(x+t) + g(x-t)) \;+\; \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

We see that the domain of influence of an interval $(x_0, 0)$ is the forward cone $\{(x,t) : x \in (x_0 - t, x_0 + t)\}$. The domain of dependence of a point (x_0,t_0) is the backward cone given by $\{(x,t) : x \in [x_0 - (t_0 - t), x_0 + (t_0 - t)]\}$.

When $n = 3$, the solution is

$$u(x,t) \;=\; \fint_{\partial B(x,t)} (g(y) \;+\; \nabla g(y)\cdot(y-x) \;+\; t h(y))\; dS(y).$$

The domain of influence of a ball $B(x_0,r)$ is $\{(x,t) : r - t < |x - x_0| < r + t\}$. The domain of dependence of a point (x_0,t_0) is the backward cone given by $\{(x,t) : |x - x_0| = t_0 - t\}$. In particular we see that if someone shouts from a point in a three dimensional world for a length of time t_1, then an observer will hear this shout for a length of time t_1, where as in a one dimensional world they might hear the effects of this shout forever. (We assume Duhamel's principle which will be discussed later.)

Solution in 2 dimensions. Suppose that $u(x_1,x_2,t)$ is a solution to the 2-dimensional wave equation (*). Set

$$\bar{u}(x_1,x_2,x_3,t) \;=\; u(x_1,x_2,t), \qquad g(x_1,x_2,x_2) = g(x_1,x_2), \qquad h(x_1,x_2,x_3) = h(x_1,x_2).$$

Then $\bar{u}$ solves the 3-dimensional wave equation with initial data $\bar{u} = \bar{g}$ and $\bar{u}_t = h$ when $t = 0$. Hence for $x = (x_1,x_2)$,

$$\bar{u}(x,x_3,t) \;=\; \fint_{\partial B_3(x,x_3,t)} \big(\bar{g}(y,y_3) \;+\; \nabla\bar{g}(y,y_3)\cdot((y,y_3)-(x,x_3)) \;+\; t\bar{h}(y,y_3)\big)\; dS_3(y,y_3).$$

To compute $u(x,t)$ we may as well set $x_3 = 0$. Now $\partial B_3(x,0,t)$ is a union of 2 graphs over $B_2(x,t)$ given by

$$y \;\rightarrow\; \pm y_3 = \pm(t^2 - |y|^2)^{1/2},$$

and

$$dS_3 \;=\; \left(1 + (\partial_{y_1} y_3)^2 + (\partial_{y_2} y_3)^2\right)^{1/2} \;=\; \left(1 + \frac{y_1^2}{t^2 - |y|^2} + \frac{y_2^2}{t^2 - |y|^2}\right)^{1/2} \;=\; \frac{t}{(t^2 - |y|^2)^{1/2}}.$$

Now

$$\nabla_3 \bar{g}(y, y_3) \;=\; (\partial_{y_1} g, \partial_{y_2} g, 0).$$

So for $y = (y, y_3)$,

$$\nabla_3 \bar{g}(y, y_3) \cdot ((y, y_3) - (x, 0)) \;=\; \nabla_2 g \cdot (y - x).$$

$$u(x,t) \;=\; \frac{2}{t^2 |\partial B_3(0,1)|} \int_{B_2(x,t)} (g(y) \;+\; \nabla g(y) \cdot (y - x) \;+\; t h(y)) \, \frac{t}{(t^2 - |y|^2)^{1/2}} \, dy.$$

Here, the factor 2 comes from the fact that the sphere $\partial B_3(x, x_3, t)$ is a union of 2 graphs over the disc $B_2(x,t)$. Now $|\partial B_3(0,1)| = 4\pi$ and $|B_2(0,1)| = \pi$. Hence

$$u(x,t) \;=\; \frac{1}{2} \fint_{B_2(x,t)} (g(y) \;+\; \nabla g(y) \cdot (y - x) \;+\; t h(y)) \, \frac{t}{(t^2 - |y|^2)^{1/2}} \, dy.$$