Lecture 21: Wave Equation in Odd Dimensions

We assume that we have a solution to

$$(*) \qquad \begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g, \qquad u_t = h & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

Recall that defining

$$\begin{aligned} U(x,r,t) &= ⨍_{\partial B(x,r)} u(y,t)\,dS(y), \\ G(x,r) &= ⨍_{\partial B(x,r)} g(y)\,dS(y), \\ H(x,r) &= ⨍_{\partial B(x,r)} h(y)\,dS(y), \end{aligned}$$

we have the *Euler-Poisson-Darboux* formula,

$$(*) \qquad \begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \text{in } \mathbb{R}_+ \times (0,\infty), \\ U = G, \quad U_t = H & \text{on } \mathbb{R}_+ \times \{0\}. \end{cases}$$

We showed that in 3 dimensions for x fixed, defining

$$\tilde{U}(r,t) = rU(x,r,t), \qquad \tilde{G}(r) = rG(x,r), \qquad \tilde{H}(r) = H(x,r),$$

then $\tilde{U}$ satisfies the 1 dimensional wave equation,

$$(**) \qquad \begin{cases} \tilde{U}_r r = \tilde{U}_t t & r \geq 0, t > 0 \\ \tilde{U} = \tilde{G}, \quad \tilde{U}_t = \tilde{H} & t = 0 \\ \tilde{U} = 0 & r = 0. \end{cases}$$

But then we can use the D'Alembert's solution to write $\tilde{U}$ in terms of $\tilde{G}$ and $\tilde{H}$. Finally, we note that

$$\lim_{r\to 0} \frac{\tilde{U}}{r} = u(x,t),$$

so we can use this to write u in terms of g and h.

The case n odd. We wish to modify this argument to deal with the general case with $n = 2k+1$ odd. Assume then that $u \in C^{(n+1)/2}(\mathbb{R}^n \times [0,\infty)$ is a solution to (*). Define U, G, H as above, but now set

$$\begin{aligned} \tilde{U} &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{(n-3)/2} (r^{n-2}U(x,r,t)), \\ \tilde{G} &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{(n-3)/2} (r^{n-2}G(x,r)), \\ \tilde{H} &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{(n-3)/2} (r^{n-2}H(x,r)), \end{aligned}$$

We claim that $\tilde{U}$ is in $C^2([0,\infty)\times[0,\infty))$ and satisfies the one dimensional wave equation (**), so that D'Alembert's solution gives

$$\tilde{U}(x,r,t) \;=\; \frac{1}{2}(\tilde{G}(r+t)-\tilde{G}(t-r)) \;+\; \frac{1}{2}\int_{t-r}^{r+t}\tilde{H}(s)\,ds, \qquad 0\le r\le t.$$

However, we will also see that

$$\lim_{r\to 0}\frac{\tilde{U}(x,r,t)}{\gamma_n r} \;=\; u(x,t),$$

where

$$\gamma_n = 1\cdot 3\cdots(n-2),$$

so as in the 3-dimensional case,

$$\begin{aligned}
u(x,t) \;&=\; \frac{1}{\gamma_n}\lim_{r\to 0}\left(\frac{\tilde{G}(r+t)-\tilde{G}(t-r)}{2r} \;+\; \frac{1}{2r}\int_{t-x}^{x+t}H(s)\,ds\right)\\
&= (\tilde{G}'(t)+\tilde{H}(t))/\gamma_n\\
&= \frac{1}{\gamma_n}\left(\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2}\left(t^{n-2}⨍_{\partial B(x,t)}g(y)\,dS(y)\right) \;+\; \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2}\left(t^{n-2}⨍_{\partial B(x,t)}h(y)\,dy\right)\right).
\end{aligned}$$

We need to justify some steps in this derivation, and also check that the solution works. First note that if $u\in C^j(\mathbb{R}^n\times[0,\infty)$, then $U\in C^j(\mathbb{R}^n\times[0,\infty)\times[0,\infty))$, since

$$U(x,r,t) \;=\; ⨍_{\partial B(0,1)}u(x+ry,t)\,dS(y),$$

and the integrand as a function of (x,r,t) is in $C^j(\mathbb{R}^n\times[0,\infty)\times[0,\infty))$. Now we want to see that $\tilde{U}$ is in $C^2([0,\infty)\times[0,\infty))$, but we claim that

Lemma 1.

$$\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}(r^{2k-1}\phi(r)) \;=\; \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j\phi}{dr^j}(r), \tag{1}$$

where

$$\beta_0^k \;=\; 1\cdot 3\cdots(2k-1).$$

To see this, let's calculate

$$\left(\frac{1}{r}\frac{d}{dr}\right)(r^{2k-1}\phi(r)) \;=\; (2k-1)r^{2k-3}\phi(r) \;+\; r^{2k-2}\phi'(r).$$

Differentiating again, and not worrying about any of the coefficients except the one for $\phi(r)$,

$$\left(\frac{1}{r}\frac{d}{dr}\right)^2 (r^{2k-1}\phi(r)) \;=\; (2k-1)(2k-3)r^{2k-5}\phi(r) + ?r^{2k-4}\phi'(r) + ?r^{2k-3}\phi^{(2)}(r).$$

Continuing like this, we conjecture

$$\left(\frac{1}{r}\frac{d}{dr}\right)^\ell (r^{2k-1}\phi(r))$$

$$= \;(2k-1)(2k-3)\cdots(2k-2\ell+1)r^{2k-2\ell-1}\phi(r) \;+\; \sum_{j=1}^{\ell}\beta_j^{\ell k} r^{2k-2\ell+j-1}\phi^{(j)}(r),$$

for some constants $\beta_j^{\ell k}$. It is easy to check the inductive step by differentiating. By setting $n = 2k+1$, so $k = (n-1)/2$, then $\beta_0^k = \gamma_n$, and we see that

$$\lim_{r\to 0}\frac{\tilde{U}(r,t)}{\gamma_n r} \;=\; \lim_{r\to 0}(U(x,r,t) + *r\partial_r U(x,r,t) + \ldots + *r^k\partial_r^{k-1}U(x,r,t)) = \lim_{r\to 0} U(x,r,t) \;=\; u(x,t),$$

where we have omitted the precise constants. Now we need to see that $\tilde{U}$ indeed satisfies the one dimensional wave equation. This follows from

Lemma 2.

$$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}(r^{2k-1}\phi(r)) \;=\; \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\phi}{dr}(r)\right).$$

We postpone the proof of this. We first note a general fact from Riemannian geometry which is easy to check in this case:

$$r^{-2k}\partial_r r^{2k}\partial_r U \;=\; U_{rr} \;+\; \frac{2k}{r}U_r.$$

Because of the Euler-Poincaré-Darboux theorem, this equals U_{tt}. Then

$$\begin{aligned}
\tilde{U}_{rr}(r,t) \;&=\; \frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}U(x,r,t)) \\
&=\; \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k}\left(r^{2k}\frac{\partial U(x,r,t)}{\partial r}\right) \\
&=\; \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1}r^{-2k}\frac{\partial}{\partial r}\left(r^{2k}\frac{\partial U(x,r,t)}{\partial r}\right) \\
&=\; \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1}U_{tt}(x,r,t) \;=\; \tilde{U}_{tt}.
\end{aligned}$$