Lecture 22: Existence for the wave equation

Theorem. *Suppose n is odd, $g \in C^{(n+3)/2}(\mathbb{R}^n)$ and $h \in C^{(n+1)/2}(\mathbb{R}^n)$. Set*

$$\gamma_n \;=\; 1 \cdot 3 \cdots (n-2),$$

and

$$u(x,t) \;=\; \frac{1}{\gamma_n}\,\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2}\left(t^{n-2}⨍_{\partial B(x,t)} g(y)\,dS(y)\right)$$
$$+\;\gamma_n\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2}\left(t^{n-2}⨍_{\partial B(x,t)} h(y)\,dS(y)\right).$$

Then u solves the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g, \qquad u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Proof. First we assume $g = 0$ and prove the result, then we assume $h = 0$ and prove the result. Assume then that $h \in C^{(n+1)/2}(\mathbb{R}^n)$, and set $k = (n-1)/2$ and

$$u(x,t) \;=\; \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(t^{2k-1}H(x,t)\right), \qquad H(x,t) \;=\; ⨍_{\partial B(x,t)} h(y)\,dS(y).$$

Lemma 2. *(from last time)*

$$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) \;=\; \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\phi}{dr}(r)\right).$$

When $k = 1$ this is easy to check. For the inductive step, assume the result holds for k. Then

$$\begin{aligned}
\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k+1}\phi(r)\right) \;&=\; \frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}r^{-2k}\frac{d}{dr}\left(r^{2k+1}\phi(r)\right)\right) \\
&= \frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\left(r\frac{d\phi(r)}{dr} \;+\; (2k+1)\phi(r)\right)\right) \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d}{dr}\left(r\frac{d\phi(r)}{dr} \;+\; (2k+1)\phi(r)\right)\right) \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k+1}\frac{d^2\phi(r)}{dr^2} \;+\; (2k+2)r^{2k}\frac{d\phi(r)}{dr}\right) \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k+1}\left(r^{2k+2}\frac{d\phi(r)}{dr}\right).
\end{aligned}$$

Now

$$\gamma_n u_{tt}(x,t) \;=\; \frac{\partial^2}{\partial t^2}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(t^{2k-1}H(x,t)\right) \;=\; \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k}\left(t^{2k}\frac{\partial}{\partial t}H(x,t)\right),$$

but

$$t^{2k}\frac{\partial}{\partial t}H(x,t) \;=\; t^{2k}\frac{\partial}{\partial_t}⨍_{\partial B(x,t)} h(y)\,dS(y) \;=\; \frac{t^{2k+1}}{n}⨍_{B(x,t)} \Delta h(y)\,dy \;=\; \frac{1}{n|B(0,1)|}\int_{B(x,t)} \Delta h(y)\,dy$$

So

$$u_{tt}(x,t) \;=\; \frac{1}{n|B(0,1)|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\frac{1}{t}\frac{\partial}{\partial t}\int_{B(x,t)} \Delta h(y)\,dy$$

$$= \frac{1}{n|B(0,1)|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\frac{1}{t}\int_{\partial B(x,t)} \Delta h(y)\,dS(y) \;=\; \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1} t^{2k-1}⨍_{\partial B(x,t)} \Delta h(y)\,dS(y)$$
$$= \gamma_n \Delta u(x,t).$$

Now for the case $h = 0$, $g \neq 0$, $g \in C^{(n+3)/2}(\mathbb{R}^n)$, define

$$u(x,t) \;=\; \frac{\partial}{\partial t}v(x,t), \qquad v(x,t) \;=\; \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(t^{2k-1}G(x,t)\right), \qquad G(x,t) \;=\; ⨍_{\partial B(x,t)} g(y)\,dS(y).$$

By the argument above $v_{tt} = \Delta v$, but the $u_{tt} = v_{ttt} = \Delta v_t = \Delta u$. To see the boundary behavior of u in both cases, we use

Lemma 1.

$$\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) \;=\; \gamma_n r\phi(r) \;+\; \sum_{j=1}^{k-1}\beta_j^k r^{j+1}\frac{d^j\phi}{dr^j}(r).$$

We just need to consider again the cases $g = 0$ and $h = 0$ separately, because we get the general behavior by summing these. Now when $g = 0$,

$$u(x,t) \;=\; \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(t^{2k-1}H(x,t)\right) \;=\; tH(x,t) + *t^2\partial_t^2 H(x,t) + \ldots + *t^k\partial_t^{k-1}H(x,t),$$

where we have omitted the precise constants. We see that

$$\lim_{t\to 0} u(x,t) \;=\; 0,$$

and

$$\lim_{t\to 0} u_t(x,t) \;=\; \lim_{t\to 0}\left(H(x,t) \;+\; *t\partial_t H(x,t) \;+\; *t^2\partial_t^2 H(x,t) \;+\; \ldots \;+\; *t^k\partial_t^k H(x,t)\right)$$
$$= \lim_{t\to 0} H(x,t) \;=\; h(x).$$

On the other hand, if $h = 0$, then

$$
\begin{aligned}
u(x,t) &= \frac{1}{\gamma_n}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(t^{2k-1}G(x,t)\right)\\
&= \partial_t\left(tG(x,t) + *t^2\partial_t G(t) + \ldots + *t^k\partial_t^{k-1}G(x,t)\right)\\
&= G(x,t) + *t\partial_t G(x,t) + \ldots *t^k\partial_t^k G(x,t).
\end{aligned}
$$

Hence

$$\lim_{t\to 0} u(x,t) = \lim_{t\to 0} G(x,t) = g(x),$$

and

$$\lim_{t\to 0} u_t(x,t) = *\lim_{t\to 0}\partial_t G(x,t),$$

but

$$
\begin{aligned}
\lim_{t\to 0}\partial_t G(x,t) &= \lim_{t\to 0}\partial_t \fint_{\partial B(0,1)} g(x+ty)\,dS(y)\\
&= \lim_{t\to 0}\fint_{\partial B(0,1)} \nabla g(x+ty)\cdot y\,dS(y) = \fint_{\partial B(0,1)}\nabla g(x)\cdot y\,dS(y) = 0,
\end{aligned}
$$

since we are integrating an odd function over the sphere.

n **even**. To solve the wave equation when n is even, as in the 2 dimensional case, we solve in one higher dimension. Set

$$\bar{g}(x_1,\ldots,x_{n+1}) = g(x_1,\ldots,x_n), \qquad \bar{h}(x_1,\ldots,x_{n+1}) = h(x_1,\ldots,x_n).$$

Then the solution to the wave equation with this boundary data is given by

$$
\begin{aligned}
u(x_1,\ldots,x_{n+1},t) &= \frac{1}{\gamma_{n+1}}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\left(t^{n-1}\fint_{\partial B((x_1,\ldots,x_{n+1}),t)}\bar{g}(y)\,dS(y)\right)\\
&\quad + \frac{1}{\gamma_{n+1}}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\left(t^{n-1}\fint_{\partial B((x_1,\ldots,x_{n+1}),t)}\bar{h}(y)\,dS(y)\right).
\end{aligned}
$$

This solution is, as we would expect, independent of x_{n+1}. It satisfies the wave equation provided $g \in C^{(n+4)/2}(\mathbb{R}^n)$ and $h \in C^{(n+2)/2}(\mathbb{R}^n)$. Our only problem is to write it in terms of g and h. Set $x = (x_1,\ldots,x_n)$. The sphere $\partial B((x,x_{n+1}),t)$ in $\mathbb{R}^{n+1}$ is a double graph over the ball $B(x,t)$ in $\mathbb{R}^n$, given by

$$y \to \pm(t^2 - |y-x|^2)^{1/2}.$$

Then on each graph,

$$
\begin{aligned}
dS(y) &= \left(1 + \left(\frac{\partial y_{n+1}}{\partial y_1}\right)^2 + \ldots + \left(\frac{\partial y_{n+1}}{\partial y_n}\right)^2\right)^{1/2} dy\\
&= \left(1 + \left(\frac{x_1-y_1}{(t^2-|y|^2)^{1/2}}\right)^2 + \ldots + \left(\frac{x_n-y_n}{(t^2-|y-x|^2)^{1/2}}\right)^2\right)^{1/2} dy\\
&= \frac{t}{(t^2-|y-x|^2)^{1/2}}\,dy
\end{aligned}
$$

Hence

(**)

$$
\begin{aligned}
u(x,t) \;=\; & \frac{2|B^n|}{\gamma_{n+1}|\partial B^{n+1}|}\left(\frac{\partial}{\partial t}\;\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\left(t^{n-1}\fint_{B(x,t)}\frac{tg(y)}{(t^2-|y-x|^2)^{1/2}}\,dS(y)\right)\right.\\
& \left. +\;\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\left(t^{n-1}\fint_{B(x,t)}\frac{th(y)}{(t^2-|y-x|^2)^{1/2}}\,dS(y)\right)\right).
\end{aligned}
$$

However, using polar coordinates at the south pole $(0,0,\ldots,-1)$ of ∂B^{n+1} gives

$$
|\partial B^{(n+1)}| \;=\; \int_0^\pi |\partial B^n|\sin^{n-1} r\,dr,
$$

and

$$
\begin{aligned}
\alpha_n \;:=\; \int_0^\pi \sin^{n-1} r\,dr \;&=\; \int_0^\pi \sin^{n-2} r\sin r\,dr\\
&=\; (n-2)\int_0^\pi \sin^{n-3} r\cos^2 r\,dr \;=\; (n-2)(\alpha_{n-2}-\alpha_n).
\end{aligned}
$$

Hence $(n-1)\alpha_n = (n-2)\alpha_{n-2}$, and

$$
\alpha_n = 2\frac{2\cdot 4\cdots(n-2)}{1\cdot 3\cdots(n-1)} \;=\; \frac{2\gamma_n}{n\gamma_{n+1}},
$$

where for n even we define

$$
\gamma_n \;=\; 2\cdot 4\cdots n.
$$

Hence we evaluate the constant

$$
\frac{2|B^n|}{\gamma_{n+1}|\partial B^{n+1}|} \;=\; \frac{2|\partial B^n|}{n\gamma_{n+1}|\partial B^{n+1}|} \;=\; \frac{1}{\gamma_n}.
$$