Lecture 23: Nonhomogeneous Wave Equation

One comment for the even dimensional existence proof. n even, $g \in C^{(n+4)/2}(\mathbb{R}^n)$ and $h \in C^{(n+2)/2}(\mathbb{R}^n)$, then

$$\begin{aligned} u(x,t) &= \frac{1}{2\cdot 4\cdots n}\left(\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\left(t^{n-1}\fint_{B(x,t)}\frac{tg(y)}{(t^2-|y-x|^2)^{1/2}}\,dS(y)\right)\right. \\ &\quad \left. + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\left(t^{n-1}\fint_{B(x,t)}\frac{th(y)}{(t^2-|y-x|^2)^{1/2}}\,dS(y)\right)\right) \end{aligned}$$

solves

$$\begin{cases} u_{tt}-\Delta u = 0 & \text{in } \mathbb{R}^n\times(0,\infty), \\ u=g, \qquad u_t = h & \text{on } \mathbb{R}^n\times\{t=0\}. \end{cases}$$

To prove this, we solve on $\mathbb{R}^{n+1}\times[0,\infty)$ with data independent of x_{n+1} and restrict to $\mathbb{R}^n\times[0,\infty)$. Set $x=(x_1,\ldots,x_n)$. In the higher dimensional solution we have terms like

$$\fint_{\partial B((x,0),t)} g(y_1,\ldots,y_n)\,dS(y_1,\ldots,y_{n+1}).$$

Considering the sphere as a double graph over the ball $B(x,t)$, we get

$$\frac{2}{t^n|\partial B^{n+1}|}\int_{B((x_1,\ldots,x_n),t)}\frac{tg(y)}{(t^2-|x-y|^2)^{1/2}}\,dy.$$

To replace $\int$ by $\fint$, we must evaluate

$$\frac{2|B^n|}{|\partial B^{n+1}|}.$$

This is a polar coordinate calculation. Indeed, using polar coordinates at the south pole $(0,0,\ldots,-1)$ of ∂B^{n+1} gives

$$|\partial B^{(n+1)}| = \int_0^\pi |\partial B^n|\sin^{n-1} r\,dr,$$

and

$$\begin{aligned} \alpha_n &:= \int_0^\pi \sin^{n-1} r\,dr = \int_0^\pi \sin^{n-2} r\sin r\,dr \\ &= (n-2)\int_0^\pi \sin^{n-3} r\cos^2 r\,dr = (n-2)(\alpha_{n-2}-\alpha_n). \end{aligned}$$

Hence $(n-1)\alpha_n = (n-2)\alpha_{n-2}$, and

$$\alpha_n = 2\frac{2\cdot 4\cdots(n-2)}{1\cdot 3\cdots(n-1)},$$

Hence we evaluate the constant

$$\frac{2|B^n|}{|\partial B^{n+1}|} = \frac{1\cdot 3\cdots(n-1)}{2\cdot 4\cdots(n-2)\cdot n}.$$

Nonhomogeneous Wave Equation.

Now we solve

$$(*)\qquad \begin{cases} u_{tt} - \Delta u = f & \text{on } \mathbb{R}^n \times [0,\infty), \\ u(x,0) = 0, \quad u_t(x,0) = 0. \end{cases}$$

We can use Duhamel's principle to solve this. Recall what Duhamel's principle says. It says that if the solution to

$$\begin{cases} U_t \ - \ LU \ = \ 0 & t > s \\ U(s) = G \end{cases}$$

is given by $S(t;s)G$, then the solution to

$$\begin{cases} U_t \ - \ LU \ = \ F(s) & t > 0 \\ U(0) = 0 \end{cases}$$

is given by

$$U(t) \ = \ \int_0^t S(t;s)F(s)\,ds.$$

Now we want to solve

$$\begin{cases} U_{tt} \ - \ LU \ = \ F(s) & t > 0 \\ U(0) = 0. \end{cases}$$

We write this as a system by setting

$$V \ = \ \begin{pmatrix} U \\ U_t \end{pmatrix}, \qquad \tilde{L} = \begin{pmatrix} 0 & 1 \\ L & 0 \end{pmatrix}.$$

Then we want to solve

$$\begin{pmatrix} U \\ U_t \end{pmatrix}_t - \begin{pmatrix} 0 & 1 \\ L & 0 \end{pmatrix}\begin{pmatrix} U \\ U_t \end{pmatrix} \ = \ \begin{pmatrix} 0 \\ F(t) \end{pmatrix}, \qquad t > 0$$

with boundary conditions

$$\begin{pmatrix} U(0) \\ U_t(0) \end{pmatrix} \ = \ 0.$$

For this, Duhamel's principle tells us we need to use the solution operator to the homogeneous problem

$$\begin{pmatrix} U \\ U_t \end{pmatrix}_t - \begin{pmatrix} 0 & 1 \\ L & 0 \end{pmatrix}\begin{pmatrix} U \\ U_t \end{pmatrix} \ = \ 0,$$

with boundary conditions

$$\begin{pmatrix} U(s) \\ U_t(s) \end{pmatrix} \ = \ \begin{pmatrix} 0 \\ F(s) \end{pmatrix}.$$

Define $u(x,t;s)$ to be the solution of

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{on } \mathbb{R}^n \times [s,\infty) \\ u(x,s;s) = 0, \quad u_t(x,s;s) = f(x,s). \end{cases}$$

Then Duhamel's principle gives a solution to (*) of the form

$$(**)\qquad u(x,t) \ = \ \int_0^t u(x,t;s)\,ds.$$

Theorem. *For $n \geq 2$, suppose $f \in C^{[n/2]+1}(\mathbb{R}^n \times [0,\infty)$ and define $u(x,t)$ by (**). Then $u(x,t)$ solves (*).*

Proof. $u(x,t;s)$ defined in (**) is in $C^2(\mathbb{R}^n \times [s,\infty))$ for each fixed s. One can also give local bounds on $u(x,t)$ and its first and second order x,t derivatives in terms of local bounds on f and its first $[n/2]+1$ derivatives.

$$u_t(x,t) \;=\; u(x,t;t) \;+\; \int_0^t u_t(x,t;s)\,ds \;=\; \int_0^t u_t(x,t;s)\,ds.$$

Then

$$u_{tt}(x,t) \;=\; u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\,ds \;=\; f(x,t) + \int_0^t \Delta u(x,t;s)\,ds \;=\; f(x,t) + \Delta u(x,t).$$

We carried out the examples on page 82 of the book.