

Lecture 24: Energy for Uniqueness

Let $U \subset \mathbb{R}^n$ be a bounded open set. For $T > 0$, set $U_T = U \times (0, T]$ and $\Gamma_T = \bar{U}_T \setminus U_T$.

Theorem. *There exists at most one solution $u(x, t) \in C^2(\bar{U})$ to the wave equation*

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } U_T, \\ u = g & \text{on } \Gamma_T, \\ u_t = h & \text{on } U \times \{0\}. \end{cases}$$

Proof. If u and \tilde{u} are two solutions, define $w = u - \tilde{u}$. Then w satisfies

$$\begin{cases} w_{tt} - \Delta w = 0 & \text{in } U_T, \\ w = 0 & \text{on } \Gamma_T, \\ w_t = 0 & \text{on } U \times \{0\}. \end{cases}$$

Define

$$e(t) = \frac{1}{2} \int_U w_t^2(x, t) + |\nabla w|^2(x, t) dx.$$

Then

$$e_t(t) = \int_U w_{tt}w_t + \nabla w_t \cdot \nabla w dx = \int_U w_{tt}w_t - w_t \cdot \Delta w dx = 0.$$

Here there are no boundary terms because $w = 0$ on $\partial U \times [0, T]$, so $w_t = 0$ on $\partial U \times [0, T]$. Since $e_t(t) = 0$, we conclude $e(t) = 0$, and hence $w_t = 0$ on $U \times [0, t]$, so $w = 0$ on \bar{U}_T .

Now we show that for any C^2 solution, we have finite speed of propagation. Put another way, consider the ball $B(x_0, t_0)$ in \mathbb{R}^n , and the cone

$$C(x_0, t_0) = \{(x, t) : 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}.$$

Theorem. *If $u \in C^2(C(x_0, t_0))$ solves*

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{on } C(x_0, t_0), \\ u = u_t = 0 & \text{on } B(x_0, t_0) \times \{0\}, \end{cases}$$

then $u = 0$ on $C(x_0, t_0)$.

Proof. For $0 \leq t \leq t_0$, set

$$e(t) = \frac{1}{2} \int_{B(x_0, t_0-t)} u_t^2(x, t) + |\nabla u|^2(x, t) dx.$$

Then

$$\begin{aligned}
e_t &= \int_{B(x_0, t_0-t)} u_t u_{tt} + \nabla u_t \cdot \nabla u \, dx - \frac{1}{2} \int_{\partial B(x_0, t_0-t)} u_t^2 + |\nabla u|^2 \, dS(x). \\
&= \int_{B(x_0, t_0-t)} u_t u_{tt} - u_t \Delta u \, dx + \int_{\partial B(x_0, t_0-t)} u_t \nabla u \cdot \nu - \frac{1}{2} u_t^2 - \frac{1}{2} |\nabla u|^2 \, dS(x) \\
(*) &= \int_{\partial B(x_0, t_0-t)} u_t \nabla u \cdot \nu - \frac{1}{2} u_t^2 - \frac{1}{2} |\nabla u|^2 \, dS(x),
\end{aligned}$$

where we have used the fact that u satisfies the wave equation. However, using Cauchy Schwarz and the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, we have

$$\begin{aligned}
\int_{\partial B(x_0, t_0-t)} u_t \nabla u \cdot \nu &\leq \left(\int_{\partial B(x_0, t_0-t)} u_t^2 \, dS(x) \right)^{1/2} \left(\int_{\partial B(x_0, t_0-t)} |\nabla u|^2 \, dS(x) \right)^{1/2} \\
&\leq \frac{1}{2} \int_{\partial B(x_0, t_0-t)} \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 \, dS(x)
\end{aligned}$$

and so we see from (*) that $e_t \leq 0$. But $e(0) = 0$ and $e(t) \geq 0$ for each t , hence $e(t) = 0$ on $[0, t_0]$, and so $u_t = 0$ on $C(x_0, t_0)$ hence $u = 0$ on $C(x_0, t_0)$.