Lecture 3: The Transport Equation.

Today we have an easy equation to solve! For $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, we are looking for a continuous function $u(x,t)$ where $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$, solving

$$u_t \ + \ b_1 u_{x_1} \ + \ b_2 u_{x_2} \ + \ \ldots \ + \ b_n u_{x_n} \ = \ 0.$$

Writing $Du = D_x u = (u_{x_1}, \ldots, u_{x_n})$, we can write this equation as

$$u_t + b \cdot Du = 0, \qquad x \in \mathbb{R}^n,\ t \in (0,\infty). \tag{1}$$

(We have distinguished the variable t from the other variables, because in a moment we will consider the initial value problem with data assigned at $t = 0$. For this purpose it is important that the coefficient in front of u_t is non-zero.)

Solving the transport equation is trivial in view of the chain rule, which says that

(2)

$$\frac{d}{ds}u(x+sb,t+s) \ = \ b_1 u_{x_1}(x+sb,t+s) + \ldots + b_n u_{x_n}(x+sb,t+s) + u_t(x+sb,t+s).$$

Hence (1) becomes

$$\frac{d}{ds}u(x+sb,t+s) \ = \ 0,$$

so $u(x+sb,t+s)$ is a constant independent of s (it may depend on x and t). In particular, the values with $s = 0$ and $s = -t$ are equal, so

$$u(x,t) \ = \ u(x-bt,0).$$

We see that the general solution to the *initial value problem*

$$\begin{cases} u_t + b \cdot Du = 0, & x \in \mathbb{R}^n,\ t \in (0,\infty) \\ u(x,0) = g(x), & x \in \mathbb{R}^n \end{cases}$$

is

$$u(x,t) \ = \ g(x-bt). \tag{3}$$

If g is C^1 the solution is classical. Otherwise (3) defines a *weak solution.*

Now we will solve the *inhomogeneous* equation

$$\begin{cases} u_t + b \cdot Du = f, & x \in \mathbb{R}^n,\ t \in (0,\infty) \\ u(x,0) = g(x), & x \in \mathbb{R}^n \end{cases}$$

In light of (2), the first equation has the form

$$\frac{d}{ds}u(x+sb,t+s) \ = \ f(x+bs,t+s).$$

so by the fundamental theorem of calculus from $s=-t$ to $s=0$ we have,

$$u(x,t) \;=\; u(x-bt,0) \;+\; \int_{-t}^{0} f(x+bs,t+s)\,ds.$$

Changing variable in the integral to $S=s+t$ for s and using the initial condition for u gives

$$u(x,t) \;=\; g(x-bt) \;+\; \int_{0}^{t} f(x+b(S-t),S)\,dS.$$

Ordinary Differential Equations

So far, our explicit solutions to partial differential equations have boiled down to explicit solutions to ordinary differential equations. In the homework you will see simple initial value problems for partial differential equations which have no solution. This is not the case for ordinary differential equations, which always have solutions, at least locally. We will prove this result. First we remark that an ordinary differential equation of any order can be rewritten as a first order *system*, that is a first order vector valued equation. Indeed, if $0\in(a,b)$ and $u:(a,b)\to\mathbb{R}$ solves

$$\begin{cases} u^{(k)}(t) \;=\; f(u^{(k-1)}(t),\ldots,u'(t),t), \qquad t\in(a,b) \\ u(0)=g_0 \\ u'(0)=g_1 \\ \vdots \\ u^{(k-1)}(0)=g_{k-1}, \end{cases}$$

this can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} u^{(k-1)} \\ u^{(k-2)} \\ \vdots \\ u \end{pmatrix}(t) \;=\; F\begin{pmatrix} u^{(k-1)}(t) \\ u^{(k-2)}(t) \\ \vdots \\ u(t) \end{pmatrix} := \begin{pmatrix} f(u^{(k-1)}(t),\ldots,u'(t),t) \\ u^{(k-1)}(t) \\ \vdots \\ u'(t) \end{pmatrix}, \qquad t\in(a,b),$$

$$\begin{pmatrix} u^{(k-1)} \\ u^{(k-2)} \\ \vdots \\ u \end{pmatrix}(0) \;=\; \begin{pmatrix} g_{k-1} \\ g_{k-2} \\ \vdots \\ g_0 \end{pmatrix}.$$

We will thus study the problem of solving a *system*. We want to find a solution $u:\mathbb{R}\to\mathbb{R}^n$ to the general ordinary differential system

$$\begin{cases} du/dt \;=\; f(u,t), \qquad t\in\mathbb{R} \\ u(0)=g, \end{cases}$$

$f : (a, b) \times \mathbb{R}^n \to \mathbb{R}^n$ is a function. It turns out that it is easy to solve this if we can solve the simpler case when f is time independent, thus we want to solve

$$\begin{cases} du/dt = f(u), & t \in \mathbb{R} \\ u(0) = g. \end{cases}$$

The geometric picture is that f is a vector field defined in $\mathbb{R}^n$, and u is a curve in $\mathbb{R}^n$ parameterized by a parameter t. The differential equation states that the curve $u(t)$ is always tangent to the vector field f. In fact it says a little more. If a particle travels with position $u(t)$ at time t, then its velocity is given by $f(u)$.

As someone in the audience pointed out, when solving the transport equation (a PDE) we were looking for a solution $u : \mathbb{R}^{n+1} \to \mathbb{R}$, whereas with the ordinary differential system we are looking for a solution $u : \mathbb{R} \to \mathbb{R}^n$. These problems are rather different.