MATH 231A PARTIAL DIFFERENTIAL EQUATIONS

Lecture 4: Ordinary Differential Equations

Theorem. *Let $U \subset \mathbb{R}^n$ be an open set and suppose that $f : U \to \mathbb{R}^n$ is bounded that is $|f| \leq L$ on U for some constant L, and suppose f satisfies the uniform Lipschitz condition*

$$|f(x,t) - f(y,t)| \;\leq\; K|x-y|, \qquad x, y \in U.$$

Then for each $g \in U$, there exists $\delta > 0$ with $(-\delta, \delta) \subset (a,b)$, such that there exists a unique continuously differentiable function $u : (-\delta,\delta) \to U$ with

$$\begin{cases} u_t \;=\; f(u), & t \in (-\delta, \delta), \\ u(0) = g. \end{cases} \tag{1}$$

Proof of the Theorem.

The idea is to change the differential equation (1) into an integral equation, since integration makes functions smoother, while differentiation does the opposite. Integrating the differential equation in (1), we get

$$u(t) \;=\; g \;+\; \int_0^t f(u(\tau))\,d\tau \qquad t \in (-\delta, \delta). \tag{2}$$

Notice that for $u \in C^1((-\delta,\delta), U)$, the function u satisfies (1) if and only if it satisfies (2). Our strategy is to define, for functions $u \in C((-\delta,\delta),U)$ with $u(0) = g$,

$$(Su)(t) \;=\; g \;+\; \int_0^t f(u(\tau))\,d\tau. \tag{3}$$

We are looking for a solution of

$$u \;=\; Su. \tag{4}$$

We start with the function $u_0 \equiv g$, and form the sequence u_0, $u_1 = Su_0$, $u_2 = S^2 u_0$, ... and see that this sequence converges to a solution u of (3). We could go ahead and check this, but it is usual to write an abstract lemma.

Contraction Mapping Lemma. . *Let (X,d) be a complete metric space, and let $S : X \to X$ be a contraction, that is there exists $\rho < 1$ such that*

$$d(Sx, Sy) \;\leq\; \rho\, d(x,y), \qquad x, y \in X.$$

Then S has a unique fixed point, that is a unique point with $Sx = x$.

Proof. First note that a fixed point of S must be unique, because if $Sx = x$ and $Sy = y$, then

$$d(x,y) \;=\; d(Sx,Sy) \;\leq\; \rho d(x,y),$$

so $d(x,y) = 0$ and $x = y$.

Now let x_0 be any point of X, and consider the sequence

$$x_0, \quad x_1 = Sx_0, \quad x_2 = S^2x_0, \quad \ldots$$

Then we want to see that this sequence is Cauchy. Indeed, for $m = 1, 2, \ldots,$

$$d(x_m, x_{m+1}) \;=\; d(S^m x_0, S^m x_1) \;\leq\; \rho^m d(x_0, x_1),$$

and so

$$\begin{aligned} d(x_m, x_{m+k}) \;&\leq\; d(x_m, x_{m+1}) \;+\; d(x_{m+1}, x_{m+2}) \;+\; \ldots d(x_{m+k-1}, x_{m+k}) \\ &\leq\; d(x_0, x_1)\left(\rho^m + \rho^{m+1} + \cdots + \rho^{m+k-1}\right) \;<\; d(x_0, x_1)\frac{\rho^m}{1-\rho} \;\to\; 0 \end{aligned}$$

as $m \to \infty$, and so x_m is a Cauchy sequence. Since X is complete, this means that it has a convergent subsequence which tends, say to x. But then since S is continuous, $x_{n+1} = Sx_n \to Sx$, and so $Sx = x$ (or to say this the long way,

$$d(x_{n+1}, Sx) \;=\; d(Sx_n, Sx) \;\leq\; d(x_n, x) \;\to\; 0, \qquad n \to \infty,$$

hence $x_n \to Sx$ and $x_n \to x$, so $Sx = x$.) $\square$

To apply this Lemma, we want our space X to be given by

$$X \;=\; \{u : (-\delta, \delta) \to U \;:\; u \text{ is continuous and } u(0) = g\}.$$

However, in order that this space should be complete, we will need U to be complete, and the open set U might not be. Hence we choose $\epsilon > 0$ so that the closed ball $\overline{B_\epsilon(g)}$ of radius ϵ around g is contained in U, and set

$$X \;=\; \{u : (-\delta, \delta) \to \overline{B_\epsilon(g)} \;:\; u \text{ is continuous and bounded and } u(0) = g\}.$$

The norm is

$$d(u, v) \;=\; \sup_{t \in (-\delta, \delta)} |u(t) - v(t)|.$$

Then X is indeed a complete metric space. The fact that (X, d) is a metric space is easy to check. We just check completeness. Indeed, if u_j is a Cauchy sequence, then the sequence $u_j(t)$ is Cauchy for each $t \in (-\delta, \delta)$, but then $u_j(t)$ converges to a limit $u(t)$ which is in $\overline{B_\epsilon(g)}$ because this ball is closed. Moreover, using the fact that u_j is Cauchy in the metric d,

$$\sup_{t \in (-\delta, \delta)} |u_j(t) - u(t)| \;\leq\; \sup_{k > j} \sup_{t \in (-\delta, \delta)} |u_j(t) - u_k(t)| \;\to\; 0,$$

as $j \to \infty$, so u_j converges to u uniformly on $(-\delta, \delta)$. But the uniform limit of continuous functions is continuous, so u is continuous and $u_j \to u$ in (X, d). Hence X is a complete metric space.

Now we need to check that S defined by (3) maps X to X and is a contraction. To show that $S : X \to X$, note that

$$|Su(t) - g| \;=\; \left| \int_0^t f(u(x,s))\,ds \right| \;\leq\; Lt.$$

Hence if we choose δ small enough so that $L\delta < \epsilon$, then

$$Su : (-\delta, \delta) \to \overline{B_\epsilon(g)}. \tag{5}$$

(For future reference, we remark that (5) holds even if we start with $u \in C((-\delta,\delta), U)$ with $u(0) = g$, that we do not need u to map into $\overline{B_\epsilon(g)}$.) To see that S is a contraction,

$$|Su(t) - Sv(t)| \;\leq\; \int_0^t |f(u(\tau)) - f(v(\tau))|\,d\tau \;\leq\; \int_0^t K|u(\tau) - v(\tau)|\,d\tau \;\leq tKd(u,v).$$

Thus provided we choose $\delta K < 1$, that is $\delta < 1/K$, the operator S is a contraction. Hence by the contraction mapping lemma we get a unique fixed point $u \in X$ with $Su = u$. Now a small technical point. $u : (-\delta,\delta) \to \overline{B_\epsilon(g)}$ is unique function satisfying (2), but we must show that in fact there is no other function $\tilde{u} : (-\delta,\delta) \to U$ which satisfies (2). To see this, just note that from (5), $S\tilde{u} \in X$, so if $S\tilde{u} = \tilde{u}$, then $\tilde{u} \in X$, so $\tilde{u} = u$. Hence we get a unique C^1 solution to (1).

References

Serge Lang, *Introduction to differentiable manifolds.* Second edition. Universitext. Springer-Verlag, New York, 2002.

Spivak, Michael, *A comprehensive introduction to differential geometry. Vol. I.* Publish or Perish 1999