MATH 231A PARTIAL DIFFERENTIAL EQUATIONS

Lecture 5: The Laplacian

From last time:

$$\begin{cases} u_t \;=\; f(u), & t \in (-\delta, \delta), \\ u(0) = g. \end{cases} \tag{1}$$

If f is continuous then solutions exist but may not be unique: Piano's existence theorem gives existence. A counterexample to uniqueness is

$$\begin{cases} u_t \;=\; |u|^{\epsilon}, & t \in (-\delta, \delta), \\ u(0) = 0. \end{cases}$$

two solutions, $u \equiv 0$, and $u = ((1-\epsilon)t)^{1/(1-\epsilon)}$.

Laplace Operator

$$\Delta u \;=\; u_{x_1 x_1} \;+\; \ldots \;+\; u_{x_n x_n}.$$

Divergence Theorem. *Suppose U is a bounded open subset of $\mathbb{R}^n$ with C^1 boundary, and let ν be the outward unit normal on ∂U. Suppose that F is a C^1 vector field on $\overline{U}$. Then*

$$\int_{\partial U} F \cdot \nu \, dS \;=\; \int_U \nabla \cdot F \, dx.$$

Here,

$$\nabla \cdot F \;=\; (F_1)_{x_1} \;+\; \ldots \;+\; (F_n)_{x_n},$$

and a C^1 boundary is a boundary which at each of its points, can locally be written as a C^1 graph over a coordinate hyperplane $x_j = 0$.

Laplace equation $\Delta u = 0$.

Poisson equation $-\Delta u = f$.

Heat equation $u_t = \Delta u$.

Physical interpretation of the heat equation. $u(x)$ represents concentration of a chemical in solution at x. $F = -\nabla u$ is the flow of the salt (from high to low concentrations) and then for a region U,

$$\frac{d}{dt} \int_U u \, dx \;=\; -\int_{\nabla U} F \cdot \nu \, dS \;=\; -\int_U \nabla \cdot F \, dx \;=\; \int_U \Delta F \, dx.$$

Hence we get the heat equation. Laplace equation is a steady state solution - no change with time. The wave equation is $u_{tt} = \Delta u$, so Laplace equation can also give a stationary wave.

Existence Theorem. *Suppose that U is a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Suppose that g is a continuous function on ∂U, and suppose that f is a bounded function on $\overline{U}$ which is Hölder continuous at each point of U. Then there exists a unique classical solution to the equation*

$$\begin{cases} -\Delta u \;=\; f, & \textit{in } U \\ u = g & \textit{on } \partial U. \end{cases}$$

We say that $f : U \to \mathbb{R}$ is *Hölder continuous with exponent* α at x, where $0 < \alpha < 1$, if there exits C such that for all $y \in U$,

$$(*) \qquad |f(x) - f(y)| \;\leq\; C|x - y|^{\alpha},$$

f is *locally Hölder continuous* on U if for each compact subset K of U there exists C such that (*) holds for all $x, y \in K$.

We will later prove the existence theorem for some nice domains U, and assuming that $f \in C^2(U)$. We break this down into two problems.
(1). Solve

$$-\Delta u \;=\; f, \qquad \text{in } U,$$

without regard to the boundary conditions.

(2). The *Dirichlet problem*:

$$\begin{cases} -\Delta u \;=\; 0, & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

Notice that if we can solve (1) and (2), then we can solve (*) by subtracting the solution to (2) where g is given by the boundary values to the solution of (1).

Today we discuss this second problem. We will make use of *convolutions*. If $\phi, \psi : \mathbb{R}^n \to \mathbb{R}$, then

$$(\phi * \psi)(x) \;:=\; \int_{\mathbb{R}^n} \phi(x-y)\psi(y)\,dy \;=\; \int_{\mathbb{R}^n} \phi(y)\psi(x-y)\,dy.$$

Then

$$\phi * \psi \;=\; \psi * \phi.$$

Now for smooth functions of compact support, convolution computes with taking the Laplacian, that is

$$\Delta(\phi * \psi) \;=\; \Delta_x \int_{\mathbb{R}^n} \phi(x-y)\psi(y)\,dy \;=\; \int_{\mathbb{R}^n} \Delta_x \phi(x-y)\psi(y)\,dy \;=\; (\Delta\phi) * \psi.$$

We will solve the equation in (1) by some magic (which can be made rigorous if we define distributions properly). Consider the (Dirac) δ function defined by

$$\delta(x) \;=\; \begin{cases} 0 & x \neq 0, \\ \infty & x = 0 \end{cases}, \qquad\qquad \int_{\mathbb{R}^n} \delta(x) \;=\; 1.$$

Then for f continuous,

$$\delta * f \;=\; f.$$

Indeed,

$$\delta * f(x) \;=\; f * \delta(x) \;=\; \int_{\mathbb{R}^n} f(x-y)\delta(y)\,dy \;=\; f(x).$$

Our problem is now reduced to solving

$$\Delta\Phi \;=\; \delta, \tag{3}$$

for then $\Phi * f$ will solve (1). Indeed,

$$\Delta(\Phi * f) \;=\; (\Delta\phi) * f \;=\; \delta * f \;=\; f.$$

Our search for a solution to (3) is made simple by noticing that δ is rotationally symmetric, and since Δ preserves rotational symmetry, we might expect to find Φ rotationally symmetric. Hence we seek a function Φ which is a function of $r = \sqrt{x_1^2 + \cdots + x_n^2}$, which solves

$$\Delta\Phi(x) \;=\; \begin{cases} 0 & x \neq 0, \\ \infty & x = 0. \end{cases}$$