Lecture 6: Fundamental Solution for the Laplacian

We seek a solution $\Phi(x) = V(|x|)$ to the equation

$$-\Delta\Phi(x) \;=\; \begin{cases} 0 & x \neq 0, \\ \infty & x = 0. \end{cases}$$

We hope to use this to solve $-\Delta u = f$, by the magic

$$-\Delta(\Phi * f) \;=\; (-\Delta\Phi) * f \;=\; \delta * f \;=\; f.$$

To apply the Laplacian to a radial function $\Phi(x) = V(|x|)$, we set $r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$, then

$$r_{x_j} \;=\; (1/2)(x_1^2 + \cdots + x_n^2)^{-1/2}\, 2x_j \;=\; \frac{x_j}{r}.$$

Hence

$$\Phi_{x_j} \;=\; V'(r)\,\frac{x_j}{r},$$

and

$$\Phi_{x_j x_j} \;=\; V''(r)\,\frac{x_j^2}{r^2} \;+\; V'(r)\left(\frac{1}{r} - \frac{x_j^2}{r^3}\right).$$

Hence

$$\Delta\Phi \;=\; V''(r) \;+\; \frac{n-1}{r}V'(r).$$

To solve $\Delta\Phi = 0$, introduce the integrating factor r^{n-1}, so

$$(r^{n-1}V')' \;=\; r^{n-1}\cdot 0 \;=\; 0,$$

and

$$r^{n-1}V' \;=\; c_n,$$

so

$$V' \;=\; \frac{c_n}{r^{n-1}},$$

and

$$V \;=\; \begin{cases} C_n r^{-(n-2)} \;+\; A_n & n \geq 3 \\ C_2 \log r \;+\; A_2, & n = 2. \end{cases}$$

We can set the constants of integration A_n equal to zero since the equation $\Delta\Phi = \delta$ is unchanged if we change Φ by adding a constant. The value of C_n however is important if we wish to obtain the correct multiple of the delta function, and we will need to find out what it is.

Definition. The *support* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is

$$\operatorname{supp} f \;=\; \overline{\{x : f(x) \neq 0\}}.$$

We write $C_c^k(\mathbb{R}^n)$ for the space of C^k functions of compact support.

Denote the ball center x and radius ε by $B(x,\varepsilon)$. It's boundary is the sphere $\partial B(x,\varepsilon)$, and by scaling arguments, this has surface volume $|\partial B(x,\varepsilon)| = \varepsilon^{n-1}|\partial B(0,1)|$.

Theorem. *If* $f \in C^2_c(\mathbb{R}^n)$ *and we define*

$$\Phi(x) \;=\; \begin{cases} -\frac{1}{2\pi}\log|x| & n=2, \\ \frac{1}{(n-2)|\partial B(0,1)|}|x|^{-(n-2)} & n \geq 3, \end{cases}$$

then $\Phi * f \in C^2(\mathbb{R}^n)$ *solves*

$$-\Delta(\Phi * f) \;=\; f.$$

Proof. The technical problem here is to show that $\Phi * f$ is differentiable. We first note that Φ is indeed integrable close to the origin. Indeed, introducing polar coordinates,

$$\int_{B(0,1)} \Phi(x)\,dx \;=\; \int_{r=0}^{1}\int_{\partial B(0,r)} \Phi(x)\,dS(x)\,dr$$
$$=\; \int_{r=0}^{1} |\partial B(0,r)|\frac{1}{(n-2)|\partial B(0,1)|}r^{-(n-2)}\,dr \;=\; \frac{1}{n-2}\int_0^1 r\,dr \;<\; \infty.$$

We then have

$$\Phi * f(x) \;=\; \int_{\mathbb{R}^n} \Phi(y)f(x-y)\,dy,$$

and

$$(\Phi * f)_{x_j}(x) \;=\; \lim_{h\to 0}\frac{(\Phi * f)(x+he_j) - (\Phi * f)(x)}{h}$$
$$=\; \lim_{h\to 0}\int_{\mathbb{R}^n}\Phi(y)\frac{f(x+he_j-y)-f(x-y)}{h}\,dy.$$

Our problem is to interchange the limit as $h \to 0$ with the integration. This is an application of the dominated convergence theorem for those who know measure theory. Indeed, by the mean value theorem and the fact that f has compact support,

$$\left|\frac{f(x+he_j-y)-f(x-y)}{h}\right| \;\leq\; \sup_z |f_{x_j}(z)| \;\leq\; C \;<\; \infty$$

and so the integrand is uniformly bounded by $C\Phi$ independent of h. However, when say $|h| < 1$, then the integrand vanishes outside a fixed compact set. Since $C\Phi$ is integrable on any compact set, we see that the integrand is bounded by an integrable function uniformly in $|h| < 1$, so we can apply the dominated convergence theorem which states that we can interchange the limit as $h \to 0$ with the integration to obtain

$$(\Phi * f)_{x_j}(x) \;=\; \int_{\mathbb{R}^n}\Phi(y)f_{x_j}(x-y)\,dy. \tag{1}$$

(From the continuity of f_{x_j} and a second application of the dominated convergence theorem, one can see that $(\Phi * f)_{x_j}$ is continuous.)

Without using measure theory, for x fixed, we define

$$p(y,h) \;=\; \begin{cases} \frac{f(x+he_j-y)-f(x-y)}{h} & h \neq 0 \\ f_{x_j}(x-y) & h = 0. \end{cases}$$

This is a continuous function. For $|h| \leq 1$, it is compactly supported. But a continuous function on a compact set is uniformly continuous, so given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|p(y,h) - p(y,0)| \;<\; \varepsilon, \qquad\qquad \text{for all } y \in \mathbb{R}^n, \quad |h| < \delta.$$

Then when $|h| < \delta$,

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \Phi(y) \frac{f(x+he_j-y)-f(x-y)}{h}\, dy \;-\; \int_{\mathbb{R}^n} \Phi(y) f_{x_j}(x-y)\, dy \right| \\ &\qquad\qquad = \left| \int_{\mathbb{R}^n} \Phi(y)(p(y,h)-p(y,0))\, dy \right| \;\leq\; \varepsilon \left| \int \Phi\, dy \right|. \end{aligned}$$

Hence we get (1). Showing that $(\Phi * f)_{x_j}$ is continuous is slightly simpler.

The same argument can be applied to take second derivatives, and we conclude that $\Phi * f \in C^2$ and

$$\Delta(\Phi * f)(x) \;=\; \int_{\mathbb{R}^n} \Phi(y) \Delta_x f(x-y)\, dy \;=\; \int_{\mathbb{R}^n} \Phi(y)(\Delta f)(x-y)\, dy.$$

Switching to y derivatives, we have

$$\Delta(\Phi * f)(x) \;=\; \int_{\mathbb{R}^n} \Phi(y) \Delta_y f(x-y)\, dy.$$

We wish to integrate by parts to put the Laplacian onto Φ, but we cannot do this because Φ has a singularity at the origin. We get around this by cutting the singularity out and applying Green's theorem.

$$\begin{aligned} \Delta(\Phi * f)(x) \;&=\; \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y) \Delta_y f(x-y)\, dy \\ &= \lim_{\varepsilon \to 0} \Bigg(\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} (\Delta\Phi)(y) f(x-y)\, dy \;+\; \int_{\partial B(0,\varepsilon)} \Phi(y) \nabla_y f(x-y) \cdot \nu\, dS(y) \\ &\qquad\qquad - \int_{\partial B(0,\varepsilon)} (\nabla\Phi)(y) \cdot \nu \; f(x-y)\, dS(y) \Bigg) \\ &= \lim_{\varepsilon \to 0} \Bigg(\int_{\partial B(0,\varepsilon)} \Phi(y) \nabla_y f(x-y) \cdot \nu\, dS(y) \;-\; \int_{\partial B(0,\varepsilon)} (\nabla\Phi)(y) \cdot \nu \; f(x-y)\, dS(y) \Bigg) \\ &= \lim_{\varepsilon \to 0} \big(\; (I) \;+\; (II) \; \big). \end{aligned}$$

Now $\nabla_y f(x-y)\cdot\nu$ is uniformly bounded by $\sup|\nabla f|$ and $|\partial B(0,\varepsilon)| = O(\varepsilon^{n-1})$ and on $B(0,\varepsilon)$, $|\Phi| = O(\varepsilon^{n-2})$. Hence (I)$= O(\varepsilon) \to 0$ as $\varepsilon \to 0$. On the other hand computing (II), ν is supposed to be the outward normal, but this is outward to the exterior of the ball, hence it points in towards the ball and on $B(0,\varepsilon)$,

$$\nabla\Phi\cdot\nu \;=\; -\frac{\partial\Phi}{\partial r} \;=\; \frac{1}{(n-2)|\partial B(0,1)|}\frac{-\partial}{\partial r}r^{-(n-2)} \;=\; \frac{1}{|\partial B(0,1)|}\varepsilon^{-(n-1)}.$$

Hence

$$(I) \;=\; -\frac{1}{|\partial B(0,1)|}\int_{\partial B(0,\varepsilon)} f(x-y)\,dS(y).$$

This is the average value of $f(x-y)$ over $y \in \partial B(0,\varepsilon)$, which tends to $f(x)$ as $\varepsilon \to 0$. $\square$