# MATH 231A PARTIAL DIFFERENTIAL EQUATIONS

## Lecture 7: Green's functions

Last time we defined

$$\Phi(x) \;=\; \begin{cases} -\frac{1}{2\pi}\log|x| & n=2, \\ \frac{1}{(n-2)|\partial B(0,1)|}|x|^{-(n-2)} & n\geq 3, \end{cases}$$

and we showed that if $f \in C_c^2(\mathbb{R}^n)$ then $\Phi * f \in C^2(\mathbb{R}^n)$ and

$$-\Delta(\Phi * f) \;=\; f.$$

The key estimates we needed were

$$\int_{\partial B(y,\varepsilon)} |\Phi(y)|\, dS(y) \;\to\; 0, \qquad \text{as } \varepsilon \to 0,$$

and

$$\int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial r}(y) f(y)\, dS(y) \;=\; \frac{1}{|\partial B(0,\varepsilon)|}\int_{\partial B(0,\varepsilon)} f(y)\, dS(y) \;\to\; f(0) \qquad \text{as } \varepsilon \to 0.$$

Now let $U \subset \mathbb{R}^n$ be a bounded open set with $C^2$ boundary.

$$\begin{cases} -\Delta u = f & \text{on } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{*}$$

Our previous solution $\Phi * f$ will solve the first equation, but will not solve the boundary condition. We seek a function $G : \bar{U} \times \bar{U} \to \mathbb{R} \cup \infty$ so that (*) is solved by

$$u(x) \;=\; \int_U G(x,y) f(y)\, dy.$$

Write

$$Lf(x) \;=\; \int_U G(x,y) f(y)\, dy.$$

To solve (*), we are trying to find an inverse for the operator $-\Delta$. In fact, we can do this by finding either a right inverse or a left inverse, that is we can solve either of the two equations

$$-\Delta L f \;=\; f \tag{1}$$

$$L(-\Delta u) \;=\; u. \tag{2}$$

Using (1), we seek $G(x,y)$ satisfying

$$\begin{cases} -\Delta_x G(x,y) \;=\; \delta(x-y) & x \in U, \\ G(x,y) = 0, & x \in \partial U. \end{cases}$$

Now $\Phi(x-y)$ solves the first equation, we hope that it equals $G(x,y)$ up to an error which is not too badly behaved. With this in mind we seek $\phi : \bar{U}\times\bar{U}\to\mathbb{R}$ which is continuous on $\bar{U}\times U$, such that

$$G(x,y) \;=\; \Phi(x-y) \;-\; \phi(x,y). \tag{3}$$

Then

$$\begin{cases} \Delta_x\phi(x,y) \;=\; 0 & x\in U, \\ \phi(x,y) = \Phi(x-y), & x\in\partial U. \end{cases} \tag{3}$$

That is, we are looking for a harmonic function on $U$ with the same boundary values as the function g(x)=$\Phi(x-y)$. (Note that $\phi(x,y)=\Phi(x-y)$ will not do because it is bad at the point $x=y$ - it is not harmonic nor even continuous there.)

Now trying to solve (2) with Green's function of the form (3), taking $r=|y|$ and assuming the boundary condition $u=0$ on $\partial U$ is satisfied, we have

$$\begin{aligned}
&= \int_U \left(\Phi(x-y)-\phi(x,y)\right)(\Delta u)(y)\,dy \\
&= \lim_{\varepsilon\to 0}\int_{U\setminus B(x,\varepsilon)} \left(\Phi(x-y)-\phi(x,y)\right)(\Delta u)(y)\,dy \\
&= \lim_{\varepsilon\to 0}\Bigg(\int_{U\setminus B(x,\varepsilon)} \left((\Delta\Phi)(x-y)-(\Delta_y\phi)(x,y)\right)u(y)\,dy \\
&\qquad + \int_{\partial U\cup\partial B(0,\varepsilon)} \left(\Phi(x-y)-\phi(x,y)\right)\frac{\partial u}{\partial r}(y)\,dS(y) \\
&\qquad - \int_{\partial U\cup\partial B(0,\varepsilon)} \frac{\partial(\Phi(x-y)-\phi(x,y))}{\partial r}u(y)\,dS(y)\Bigg) \\
&= -\int_{U\setminus B(x,\varepsilon)} \Delta_y\phi(x,y)\,u(y)\,dy \\
&\qquad + \int_{\partial U} \left(\Phi(x-y)-\phi(x,y)\right)\frac{\partial u}{\partial r}(y)\,dS(y) \\
&\qquad - \; u(x),
\end{aligned}$$

and so this gives $-u(x)$ provided

$$\begin{cases} \Delta_y\phi(x,y) \;=\; 0 & y\in U, \\ \phi(x,y) = \Phi(x-y), & y\in\partial U. \end{cases} \tag{4}$$

We see that (3) and (4) are the same if $\phi(x,y)=\phi(y,x)$. It turns out that this is the case, indeed there is an argument which shows that $G(x,y)=G(y,x)$. We will not try to make all of this rigorous, but instead proceed to compute Green's function by symmetry arguments for some nice regions.

**Green's function for the half space.**

$$\mathbb{R}^n_+ \;=\; \{x = (x_1, \ldots, x_n) : x_n > 0\}.$$

If $x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n_+$, then the *reflection of* $x$ in $\partial\mathbb{R}^n_+$ is $\tilde{x} = (x_1, \ldots, x_{n-1}, -x_n)$. We claim that

$$\phi(x, y) \;=\; \Phi(x - \tilde{y})$$

solves (2). To see it satisfies the boundary condition, note that $\Phi(x)$ only depends on $|x|$, and reflection preserves the distance between points, that is $|\tilde{x} - \tilde{y}| = |x - y|$. Hence if $x \in \partial\mathbb{R}^n_+$, then $x = \tilde{x}$.

$$\Phi(x - \tilde{y}) \;=\; \Phi(\tilde{x} - \tilde{y}) \;=\; \Phi(x - y).$$

Hence Green's function for the half space is

$$G(x, y) \;=\; \Phi(x - y) \;-\; \Phi(x - \tilde{y}) \;=\; \frac{1}{(n-2)|\partial B(0,1)|}\left(\frac{1}{|x - y|^{n-2}} - \frac{1}{|x - \tilde{y}|^{n-2}}\right).$$

**Green's function for the ball.**

$$B^n \;=\; B^n(0, 1) \;=\; \{x \in \mathbb{R}^n : |x| < 1\}.$$

For $x = (x_1, \ldots, x_n) \in B^n$, set $x^* = (x_1, \ldots, x_{n-1}, x_n)$. Then we claim

$$G(x, y) \;=\; \Phi(x - y) \;-\; \Phi(|x|(x^* - y)).$$

Clearly for $x \in B^n$ the correction function $\phi(x, y) = \Phi(|x|(x^* - y))$ satisfies $\Delta_y \phi(x, y) = 0$ on $B^n$. We must check that it has the correct boundary values. Indeed, we claim that

$$|x - y| \;=\; |x||x^* - y|, \qquad\qquad \text{when } |y| = 1.$$

This is easy to check by drawing a picture, since when $|y| = 1$, a reflection takes $x$ to $|x|y$ and $y$ to $|x|x^* = x/|x|$, and reflections preserve distance.

Next time we will show that the solution of the problem

$$(**) \qquad \begin{cases} \Delta u = 0, & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

is given by

$$u(x) \;=\; \int_{\partial U} -\frac{\partial G(x, y)}{\partial \nu_y}\, g(y)\, dS(y).$$

We see that there is a close relationship between solving (*) and (**).