MATH 231A PARTIAL DIFFERENTIAL EQUATIONS

Lecture 8: The Poisson Kernel

Last time:

We studied the problem

$$\begin{cases} \Delta u = f, & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{1}$$

We looked for a solution by correcting the convolution solution on $\mathbb{R}^n$,

$$G(x,y) \;=\; \Phi(x-y) \;-\; \phi(x,y),$$

so that

$$u(x) \;=\; \int_U G(x,y) f(y)\, dy.$$

The Green function G, if it exists, is smooth on $U \times U$ away from the diagonal $\{(x,y) : y = x\}$, and C^1 on $\bar{U} \times \bar{U}$ away from the diagonal, and satisfies

$$\begin{cases} \Delta_y G(x,y) \;=\; 0 & y \in U, \\ G(x,y) \;=\; 0, & y \in \partial U, \\ \lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} |G(x,y)|\, dS(y) \;=\; 0, & x \in U, \\ \lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \frac{\partial G(x,y)}{\partial \nu_y} f(y)\, dS(y) \;=\; f(x), & \varepsilon \to 0, \quad x \in U. \end{cases}$$

where ν_y is the inward pointing normal on the sphere $\partial B(x,\varepsilon)$. Now if this Green's function exists, we will show that the solution of

$$\begin{cases} \Delta u = 0, & \text{in } U \\ u = g & \text{on } \partial U \end{cases} \tag{2}$$

is given by

$$u(x) \;=\; \int_{\partial U} -\frac{\partial G(x,y)}{\partial \nu_y}\, g(y)\, dS(y). \tag{*}$$

The kernel

$$P(x,y) \;=\; \frac{-\partial G(x,y)}{\partial \nu_y}, \qquad x \in U, \quad y \in \partial U$$

is called the *Poisson kernel* of U.

By summing the solutions of (1) and (2), we get the complete solution of the Poisson problem

$$\begin{cases} \Delta u = f, & \text{in } U \\ u = g & \text{on } \partial U, \end{cases} \tag{3}$$

namely

$$u(x) \;=\; \int_U G(x,y)f(y)\,dy \;+\; \int_{\partial U} -\frac{\partial G(x,y)}{\partial \nu_y}\,g(y)\,dS(y).$$

Now we check (*). Actually we will only check that if we have a solution to (2), then it is given by (*). Suppose then that u solves (2). Then by Green's theorem,

$$\begin{aligned}&\int_{U\setminus B(x,\varepsilon)} \Delta_y G(x,y)\,u(y)\,dy \;-\; \int_{U\setminus B(x,\varepsilon)} G(x,y)\,(\Delta u)(y)\,dy\\ &= \int_{\partial U\cup\partial B(x,\varepsilon)} \frac{\partial G(x,y)}{\partial \nu_y}\,u(y)\,dS(y) \;-\; \int_{\partial U\cup\partial B(x,\varepsilon)} G(x,y)\frac{\partial u}{\partial \nu}\,dS(y).\end{aligned}$$

However, the left hand side vanishes because $\Delta_y G(x,y) = \Delta u = 0$ on $U\setminus B(x,\varepsilon)$, and the second term on the right tends to zero as $\varepsilon\to 0$ since $G(x,y)=0$ for $y\in\partial U$ and the integral over $\partial B(x,\varepsilon)$ tends to zero as $\varepsilon\to 0$. Hence we are left with

$$0 = \lim_{\varepsilon\to 0}\int_{\partial U\cup\partial B(x,\varepsilon)} \frac{\partial G(x,y)}{\partial \nu_y}\,u(y)\,dS(y) \;=\; \int_{\partial U} \frac{\partial G(x,y)}{\partial \nu_y}\,u(y)\,dS(y) \;+\; u(x).$$

This completes the proof.

Poisson Kernel on B^n. Following Evans page 40 (41)-(44), we compute on B^n,

$$P(x,y)) \;=\; \frac{-\partial G(x,y)}{\partial \nu_y} \;=\; \frac{1-|x|^2}{(n-2)|\partial B^n|}\frac{1}{|x-y|^n}. \tag{4}$$

Theorem. *If $g\in C(\partial B^n)$, then the function u on B^n defined by*

$$u(x) \;=\; \int_{\partial B^n} P(x,y)\,g(y)\,dS(y) \tag{5}$$

is in $C^\infty(B^n)$ and satisfies

$$\begin{cases}\Delta u = 0, & \text{in } U\\ \lim_{x\to x_0} u(x) = g(x_0) & x_0\in U\end{cases}.$$

Proof. First, notice that $P(x,y)\in C^\infty(B^n\times\partial B^n)$. There are no singularities in the integral (5) when x is a point in B^n since the integration is over the boundary ∂B^n. Hence we can differentiate under the integral sign as many times as we like, and we see that $u\in C^\infty(B^n)$. We have

$$\Delta u(x) \;=\; \int_{\partial B^n} \Delta P(x,y)\,g(y)\,dS(y) \;=\; \int_{\partial B^n} -\frac{\partial \Delta_x G(x,y)}{\partial \nu_y}\,g(y)\,dS(y) \;=\; 0.$$

Here we have interchanged the operation Δ_x with ∂_{ν_y} and used the fact that $\Delta_x G(x,y) = 0$ for $x\in B^n$. Next time we will obtain the boundary values.