Lecture 9: Harmonic functions

Last Time: We derived the Poisson kernel on the ball:

$$P(x,y)) = \frac{1-|x|^2}{|\partial B^n|}\frac{1}{|x-y|^n}.$$

Theorem. *If $g \in C(\partial B^n)$, then the function u on B^n defined by*

$$u(x) = \int_{\partial B^n} P(x,y)\, g(y)\, dS(y)$$

is in $C^\infty(B^n)$ and satisfies

$$\begin{cases} \Delta u = 0, & \text{in } U \\ \lim_{x\to x_0} u(x) = g(x_0) & x_0 \in U. \end{cases}$$

We proved $\Delta u = 0$. It remains to see the behavior of $u(x)$ as x approaches the boundary ∂B^n. We will see that as $x \to x_0 \in \partial B^n$, as a function of $y \in \partial B^n$ the kernel $P(x,y)$ is approaching the delta function at x_0. To see this we will need to know an important property of harmonic functions, so we start by deriving this. We will use the fact that

$$|B(0,r)| = \int_0^r |\partial B(0,s)|\, ds = |\partial B(0,1)| \int_0^r s^{n-1}\, ds = \frac{r^n|\partial B(0,1)|}{n} = \frac{r|\partial B(0,r)|}{n}.$$

Define

$$\fint_{\partial B(x,r)} u(y)\, dS(y) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\, dS(y),$$
$$\fint_{B(x,r)} u(y)\, dy = \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\, dy.$$

Proposition. *Suppose that $u \in C^2(U)$ where $U \subset \mathbb{R}^n$ is open. Then for $\overline{B(x,r)} \subset U$, we have*

$$\frac{d}{dr}\fint_{\partial B(x,r)} u(y)\, dS(y) = \frac{r}{n}\fint_{B(x,r)} \Delta u(y)\, dy.$$

Proof. We apply the divergence theorem.

$$\frac{d}{dr}\fint_{\partial B(x,r)} u(y)\, dS(y) = \frac{d}{dr}\fint_{\partial B(0,1)} u(x+ry)\, dS(y)$$
$$= \fint_{\partial B(0,1)} y\cdot(\nabla u)(x+ry)\, dS(y) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} \nu\cdot(\nabla u)(y)\, dS(y) =$$
$$= \frac{r}{n|B(x,r)|}\int_{B(x,r)} \Delta u(y)\, dy.$$

The mean value property. If $u \in C^2(U)$ is harmonic in U, then

$$\begin{aligned} \fint_{\partial B(x,r)} u(y)\, dS(y) \;&=\; u(x) \\ \fint_{B(x,r)} u(y)\, dy \;&=\; u(x) \end{aligned}$$

for every ball $\overline{B(x,r)} \subset U$.

This follows easily from the proposition, since defining

$$\phi(r) \;=\; \fint_{\partial B(x,r)} u(y)\, dS(y)$$

we have $\phi(0) = u(x)$ and $\phi'(r) \equiv 0$, so $\phi(r) \equiv u(x)$. This shows that

$$\fint_{\partial B(x,r)} u(y)\, dS(y) \;=\; u(x).$$

However, using this result and polar coordinates,

$$\fint_{B(x,r)} u(y)\, dy \;=\; \frac{1}{|B(x,r)|} \int_{s=0}^{r} \int_{\partial B(x,s)} u(y)\, dS(y) ds \;=\; \frac{1}{|B(x,r)|} \int_0^r |\partial B(x,s)| u(x) ds \;=\; u(x).$$

Converse to the mean value property. If $u \in C^2(U)$ satisfies

$$\fint_{B(x,r)} u(y)\, dy \;=\; u(x)$$

for every ball $\overline{B(x,r)} \subset U$, then u is harmonic on U.

This also follows from the proposition, for if there exists a point x such that $\Delta u(x) \neq 0$, say $u(x) > 0$, then by continuity we can find a ball $B(x,R)$ with $R > 0$ on which $u > 0$, and then by the proposition,

$$\frac{d}{dr} \left(\fint_{\partial B(x,r)} u(y)\, dS(y) \right) \;>\; 0$$

for $r < R$, but then since the value at $r = 0$ is $u(x)$ we have

$$\fint_{\partial B(x,R)} u(y)\, dS(y) \;>\; u(x)$$

which is a contradiction.

Now we are ready to complete the proof of the boundary behavior of

$$\int_{\partial B^n} P(x,y)\, g(y)\, dy.$$

We claim firstly that for $x \in B^n$, defining

$$v(x) \;:=\; \int_{\partial B^n} P(x,y)\, dy \;=\; (1-|x|^2) \fint_{\partial B^n} \frac{1}{|x-y|^n}\, dS(y),$$

we have $v(x) \equiv 1$. Indeed,

$$v(0) \;=\; \fint_{\partial B^n} 1\, dS(y) \;=\; 1.$$

Furthermore, $v(x)$ is a harmonic function for $x \in B^n$. We claim that it is rotationally invariant, that is $v(x)$ depends only on $|x|$. This is true since for a rotation O,

$$\int_{\partial B^n} \frac{1}{|x-y|^n}\, dS(y) \;=\; \int_{\partial B^n} \frac{1}{|Ox-Oy|^n}\, dS(y) \;=\; \int_{\partial B^n} \frac{1}{|Ox-Y|^n}\, dS(Y),$$

where the second equality holds by making the change of variables $Y = Oy$. But then if $|x| = r$, then applying the mean value property,

$$v(x) \;=\; \fint_{B(0,r)} u(y)\, dy \;=\; v(0) \;=\; 1.$$

Now suppose that $x_0 \in \partial B^n$. Then

$$(1)\quad u(x) - g(x_0) \;=\; \int_{\partial B^n} P(x,y) g(y)\, dy - g(x_0) \;=\; \int_{\partial B^n} P(x,y)(g(y) - g(x_0))\, dy.$$

We wish to show that this tends to zero as $x \to x_0$. Our strategy is to split the domain of integration ∂B^n into two pieces, J and K. Since g is continuous, for each $\varepsilon > 0$ fixed, we can choose $\delta > 0$ so that

$$(2)\qquad |g(y) - g(x_0)| < \varepsilon \quad \text{when} \quad |y - x_0| < \delta.$$

Then pick J to be those points of ∂B^n with $|y - x_0| < \delta$, and pick K to be those points with $|y - x_0| \geq \delta$. Since $P(x,y)$ is positive and integrates to 1,

$$\left| \int_J P(x,y)(g(y) - g(x_0))\, dS(y) \right| \;\leq\; \varepsilon.$$

However, for $y \in K$, $|y - x_0| \geq \delta$, and so choosing

$$(3)\qquad |x - x_0| < \delta/2,$$

we have $|y - x| \geq \delta/2$, and so

$$P(x,y) \;\leq\; \frac{2^n(1-|x|^2)}{|\partial B^n|\ \delta^n}.$$

By choosing $|x|$ very close to 1 so that

$$1 - |x|^2 \;\leq\; \frac{\varepsilon\delta^n}{2^{n+1}\sup|g|}, \tag{4}$$

we get

$$P(x,y) \;\leq\; \frac{\varepsilon}{|\partial B^n|\ 2\sup|g|}.$$

and

$$\left|\int_K P(x,y)(g(y) - g(x_0))\,dS(y)\right| \;\leq\; \varepsilon.$$

This shows that (1) is bounded by 2ε under conditions (2),(3),(4), so $u(x) \to g(x_0)$ as $x \to x_0$.