

**Selected solutions to Homework 2.**

4. We proved that if  $U \subset \mathbb{R}^n$  is an open set and if  $f : U \rightarrow \mathbb{R}$  is a Lipschitz function, then for each point  $g \in U$  there exists a unique function  $u \in C^1((-\delta, \delta), U)$  satisfying

$$(1) \quad \begin{cases} u_t = f(u) & t \in (-\delta, \delta) \\ u(0) = g. \end{cases}$$

(a). Show that if  $f \in C^k(U)$  then  $u \in C^{k+1}((-\delta, \delta))$ .

**Solution 4 (a).** Since  $u$  satisfies

$$\frac{du}{dt} = f(u(t)),$$

the result follows from the following more general Lemma.

**Lemma.** *If  $U \subset \mathbb{R}^m$  and  $V \subset \mathbb{R}^n$  are open sets and if  $u \in C^k(U, V)$  and  $v \in C^k(V, \mathbb{R})$ , then the composition satisfies  $v \circ u \in C^k(U, \mathbb{R})$ .*

*Proof.* We show this by induction on  $k$ . Our inductive for  $k \geq 1$  is

If  $u \in C^k(U, V)$  and  $v \in C^k(V, \mathbb{R})$  then  $v \circ u \in C^k(U, \mathbb{R})$ , and for  $|\alpha| = k$ ,

$$\partial^\alpha (u \circ v)(x) = \sum_{j=1}^k \sum_{|\beta|, |\gamma^1|, \dots, |\gamma^j| \leq k} c_{\beta, \gamma^1, \dots, \gamma^j} (\partial^\beta u)(v(x)) (\partial^{\gamma^1} v)(x) \dots (\partial^{\gamma^j} v)(x).$$

The sum here is over multiindices  $\beta = (\beta_1, \dots, \beta_n)$  and  $\gamma^i = (\gamma_1^i, \dots, \gamma_m^i)$ .

The case  $k = 1$  of the Lemma follows from the chain rule, and the inductive step follows by the chain and product rule. Having proved the Lemma, we can prove 4(a) by induction. When  $k = 1$ , we see that  $f$  and  $u$  are of class  $C^1$  and hence  $f \circ u$  is of class  $C^1$ , but by the equation we see that  $du/dt$  is of class  $C^1$  and hence of class  $C^2$ . The inductive step is similar.

(b). Modify the proof of (1) to show that  $\delta$  can be chosen to be uniform on a neighborhood of the initial condition  $g$ , that is, with the same conditions as above, if  $\overline{B(x, 2\epsilon)} \subset U$ , then there exists  $\delta > 0$  such that for every  $g \in B(x, \epsilon)$  there is a unique function  $u \in C^1((-\delta, \delta), U)$  satisfying (1).

**Solution 4 (b).** In fact, no modification is necessary. If  $\overline{B(x, 2\epsilon)} \subset U$ , then whenever  $g \in B(x, \epsilon)$ , we have  $\overline{B(g, \epsilon)} \subset U$ , so by taking  $\delta < \epsilon/(\sup |f|)$  and  $\delta < 1/K$ , we get the unique solution to (1) obtained before.

5. Let  $f \in C(\mathbb{R}^n)$  be radial, that is

$$f(x) = F(|x|)$$

for some  $F \in C(\mathbb{R})$ . By integrating an ordinary differential equation in the variable  $r$ , find the unique radial function  $u \in C^2(\mathbb{R}^n)$  which solves

$$\begin{cases} \Delta u = f, \\ u(0) = 0, \end{cases}$$

and explain why it is indeed twice differentiable.

The solution is given by

$$u(x) = \int_0^{|x|} \frac{1}{R^{n-1}} \int_0^R \rho^{n-1} F(\rho) d\rho dR.$$

That is,  $u(x) = v(|x|)$  where

$$v(r) = \int_0^r \frac{1}{R^{n-1}} \int_0^R \rho^{n-1} F(\rho) d\rho dR.$$

Now it is easy to see by the fundamental theorem of calculus that  $v \in C^2[0, \infty)$ . But the map  $x \rightarrow |x|$  is in  $C^\infty(\mathbb{R} \setminus \{0\}, (0, \infty))$ . Hence the composition  $u(x) = v(|x|) \in C^2(\mathbb{R} \setminus \{0\})$ . Also we see that since  $x \rightarrow |x|$  and  $v(r)$  are continuous at zero, so is  $u(x)$ . The problem then is to check the differentiability of  $u(x)$  at  $x = 0$ . First note that applying the chain rule, away from  $x = 0$ ,

$$u_{x_j} = \frac{x_j}{|x|^n} \int_0^{|x|} \rho^{n-1} F(\rho) d\rho.$$

This is bounded by

$$\frac{|x_j|}{|x|^n} |x|^n \sup_{\rho \in [0, |x|]} |F(\rho)|$$

which tends to zero as  $x \rightarrow 0$ . By an appropriate form of the mean value theorem (or by explicit estimation) or l'hôpital's rule, we see that

$$\lim_{h \rightarrow 0} \frac{u(he_j) - u(0)}{h} = 0.$$

Hence  $u_{x_j}$  exists and is continuous at  $x = 0$ . We could write this last step as a Lemma:

Suppose that  $U$  is open in  $\mathbb{R}^n$  and  $f : U \rightarrow \mathbb{R}$  is continuous. Suppose in addition that  $f_{x_j}$  exists and is continuous at  $U \setminus y$ , where  $y$  is a point of  $U$ , and suppose that  $\lim_{x \rightarrow y} f_{x_j}(x)$  exists. Then in fact,  $f_{x_j}$  exists and is continuous on  $U$ .

Now we must check second derivatives. We have

$$\begin{aligned} u_{x_j x_k} &= \frac{\delta_{jk}}{|x|^n} \int_0^{|x|} \rho^{n-1} F(\rho) d\rho - \frac{n x_j x_k}{|x|^{n+2}} \int_0^{|x|} \rho^{n-1} F(\rho) d\rho + \frac{x_j x_k}{|x|^2} F(|x|) \\ &= \frac{\delta_{jk}}{n} F(0) + \frac{\delta_{jk}}{|x|^n} \int_0^{|x|} \rho^{n-1} (F(\rho) - F(0)) d\rho + \frac{n x_j x_k}{|x|^{n+2}} \int_0^{|x|} \rho^{n-1} (F(|x|) - F(\rho)) d\rho \end{aligned}$$

We claim that the second and third terms on the right hand side tend to zero as  $x \rightarrow 0$ . Indeed,

$$\left| \frac{\delta_{jk}}{|x|^n} \int_0^{|x|} \rho^{n-1} (F(\rho) - F(0)) d\rho \right| \leq \frac{1}{|x|^n} |x|^n \sup_{\rho < |x|} |F(\rho) - F(0)| \rightarrow 0,$$

and similarly

$$\left| \frac{nx_j x_k}{|x|^{n+2}} \int_0^{|x|} \rho^{n-1} (F(|x|) - F(\rho)) d\rho \right| \leq \sup_{\rho < |x|} |F(|x|) - F(\rho)| \rightarrow 0.$$

As before, since  $u_{x_j}$  exists and is continuous everywhere, and  $u_{x_j x_k}$  exists and is continuous at all points except 0, and has the limit  $\delta_{jk} F(0)/n$  at 0, by the little lemma used before,  $u_{x_j x_k}$  exists and is continuous at  $x = 0$ .