MATH 231B. HOMEWORK 1. DUE LAST LECTURE.

Section 5.10 problems: 2, 4, 7, 8, 13 and the following problems which came up in lectures:

H1. Suppose that X is an infinite dimensional vector space with two norms $\| \|_1$ and $\| \|_2$. Suppose that for every $v \in X$ we have

$$(*) \qquad \|v\|_1 \ \leq \ \|v\|_2.$$

If X_i is the completion of X in the norm $\| \|_i$, then there is a natural inclusion

$$X_2 \ \subset \ X_1,$$

and (*) continues to hold.

H2. Assuming the Sobolev inequalities, show that in dimension $n = 2$, if $k \geq 2$ then $H^k(M)$ is a Banach Algebra, *i.e.* there exists C such that

$$\|uv\|_{H^k(M)} \ \leq \ C\|u\|_{H^k(M)}\|v\|_{H^k(M)}.$$

H3. Suppose that $v \in L^1_{\text{loc}}(U)$ and

$$\int_U v\phi \, dx \ = \ 0 \qquad \text{for every } \phi \in C_c^\infty(U).$$

Show that $v = 0$.

H4. Suppose $\chi \in C_c^\infty(\mathbb{R})$ with $\int \chi = 0$. Show that there exists $\tilde{\chi} \in C_c^\infty(\mathbb{R})$ with $\chi = D\tilde{\chi}$.

H5. Show that if $U \subset \mathbb{R}^n$ and if $u \in L^1_{\text{loc}}(U)$ has weak derivatives $D_{x_i}u = 0$ for $i = 1, \ldots, n$, then u is constant.

H6. Show that if $U \subset \mathbb{R}^n$ is open and $u \in C(U)$ is C^1 on $\{x_n \geq 0\} \cap U$ and $\{x_n \leq 0\}$, then u has weak first order partial derivatives on U.

The following exercises are optional!

Exercise. Check that the function

$$\psi(x) \ = \ \begin{cases} e^{-1/t}, & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is smooth.

Exercise. $C_c(U)$ is dense in $L^p(U)$.

Hint: Approximate L^p functions by step functions and approximate step functions by continuous functions and approximate continuous functions by smooth functions.

Exercise: Partition of Unity Theorem. If $U \subset \mathbb{R}^n$ is an open set $\{V_i : i = 1, 2, \ldots\}$ is an open cover of U with $V_i \subset U$ for all i, then there exists a collection of functions $\zeta_i \in C^\infty(U)$ such that

$$\operatorname{supp} \zeta_i \;\subset\; V_i,$$

and

$$\sum \zeta_i = 1,$$

the sum on the left being locally finite.