MATH 231B. HOMEWORK 2. DUE LAST LECTURE.

H1. Show that if H is a Hilbert space, A is a bounded operator on H and B is a compact operator, then AB and BA are compact.

H2. Suppose that (M, g) is a smooth compact surface with Riemannian metric g. Write Δ for the Laplacian on M, and A for the area of M. For $x, y \in M$, let $d(x, y)$ denote the distance between x and y, that is the shortest possible length of a curve joining x to y. Set

$$U \;=\; M \times M \setminus \{(x,x) : x \in M\}.$$

A **Green function** on (M, g) is a smooth function

$$G : U \;\to\; \mathbb{R}$$

which satisfies the following properties:

(1) $$\Delta_x G(x,y) \;=\; -\frac{1}{A} \qquad \text{when } x \neq y.$$

(2) $$G(x,y) = G(y,x).$$

(3) $$G(x,y) \;+\; \frac{1}{2\pi}\log d(x,y) \qquad \text{is bounded .}$$

(4) $$\int_M G(x,y)\, dA(x) \;=\; 0.$$

We will assume the following

Theorem. *If $f \in C^\infty(M)$ then there exists $u \in C^\infty(M)$ such that*

$$\Delta u \;=\; f \;-\; \frac{1}{A}\int_M f\, dA,$$

and if the Green function exists then

$$u(x) \;=\; \int_M G(x,y) f(y)\, dA(y).$$

(a). Show that if $G(x, y)$ exists then it is unique.

Now suppose that $\phi \in C^\infty(M)$ and consider the metric $\tilde{g} = e^{2\phi} g$ which has Laplacian $\tilde{\Delta} = e^{-2\phi}\Delta$, area element $d\tilde{A} = e^{2\phi} dA$, and distance $\tilde{d}(x, y)$ which satisfies

$$e^{\min\phi} d(x,y) \;\le\; \tilde{d}(x,y) \;\le\; e^{\max\phi} d(x,y).$$

(b). If the Green function $G(x, y)$ exists for (M, g), then find $f \in C^\infty(M)$ such that

$$\tilde{G}(x,y) \;=\; G(x,y) \;+\; f(x) \;+\; f(y)$$

is a Green function for $(M, \tilde{g})$.