

Lecture 12. Extension.

Extension Theorem. Assume U is bounded and ∂U is C^1 , and suppose V is open with $U \subset\subset V$. Then there exists a bounded linear operator

$$E : W^{1,p}(U) \rightarrow W^{1,p}(\mathbb{R}^n)$$

such that for each $u \in W^{1,p}(U)$,

- (i) $Eu = u$ a.e. in U .
- (ii) Eu has support within V .
- (iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$ where the constant C depends only on p, U, V .

We call Eu an extension of u to \mathbb{R}^n .

Proof. First we consider a model case where $U = \{x : x_n > 0\}$, and setting $B = B(0, 1)$, we want to extend functions $u \in W^{1,p}(U)$ to $U \cup B$. Set

$$\begin{cases} B^+ = B \cap \{x_n \geq 0\}, \\ B^- = B \cap \{x_n \leq 0\}. \end{cases}$$

Assuming u is C^1 on B^+ , extend u to points $x \in B^-$ by setting

$$u^-(x) = -3u(x_1, \dots, x_{n-1}, -x_n) + 4u(x_1, \dots, x_{n-1}, -x_n/2).$$

First claim: $u = u^-$ agree on $\{x_n = 0\}$. Also $u_{x_j} = u_{x_j}^-$ on $\{x_n = 0\}$ for $j < n$. Indeed, when $x_n = 0$ we get $u^-(x) = -3u(x) + 4u(x) = u(x)$, and similarly for $u_{x_j}^-$.

Second claim: $u_{x_n} = u_{x_n}^-$ on $\{x_n = 0\}$. Indeed,

$$\frac{\partial u^-}{\partial x_n}(x_1, \dots, x_n) = 3 \frac{\partial u}{\partial x_n}(x_1, \dots, x_{n-1}, -x_n) - 2 \frac{\partial u}{\partial x_n}(x_1, \dots, x_{n-1}, -\frac{x_n}{2}).$$

Hence setting

$$Eu(x) = \begin{cases} u(x) & x \in U \\ u^-(x) & x \in B^-, \end{cases}$$

we see that $Eu|_B \in C^1(B)$. Moreover,

$$\|u^-(x)\|_{L^p(B^-)} \leq 3\|u\|_{L^p(B^+)} + 4 \left(\int_{B^+} |u(x_1, \dots, x_{n-1}, -x_n/2)|^p dx \right)^{1/p} \leq (3+4 \cdot 2^{1/p})\|u\|_{L^p(B^+)}.$$

Estimating the derivatives similarly gives

$$\|Eu\|_{W^{1,p}(U \cup B)} \leq C\|u\|_{W^{1,p}(U)}, \quad C = 3 + 2^{2+1/p}.$$

Remark. In fact we see that if $u \in W^{2,p}(U)$ and $u|_B \in C^2(B)$ and $j < n$ and $k \leq n$ then $u_{x_j x_k}^- = u_{x_j x_k}$ on $\{x_n = 0\}$. However, $u_{x_n x_n}^- = -2u_{x_n x_n}$. Nevertheless it is easy to see that $Eu_{x_n x_n}$ does exist in the weak sense, and

$$\|Eu\|_{W^{2,p}(U \cup B)} \leq C\|u\|_{W^{2,p}(U)}.$$

Exercise. Show that if $U \subset \mathbb{R}^n$ is open and $u \in C(U)$ is C^1 on $\{x_n \geq 0\} \cap U$ and $\{x_n \leq 0\}$, and then u has weak first order partial derivatives on U .

Now for the general result we follow 5.4 steps 5-7.