Lecture 14. Sobolev Inequalities.

Left over from last time:

Lemma. *If* $u \in W^{1,p}(\{x_n \geq 0\})$ *and* $Tu = 0$ *on* $\{x_n = 0\}$ *then*

$$\int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p \, dx' \;\leq\; C x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du|^p \, dx' dt.$$

Exercise. If $u \in W^{1,p}(\mathbb{R}^n_+)$ then there exists a sequence of functions $u_m \in C^1(\overline{\mathbb{R}^n_+})$ with $u_m \to u$ in $W^{1,p}(\mathbb{R}^n_+)$.

Proof of the Lemma. Choose such a sequence $u_m \to u$. Then

$$u_m(x', x_n) \;\leq\; u_m(x', x_n) \;+\; \int_0^{x_n} |u_{m,x_n}(x', t)| \, dt.$$

Thus

$$\int_{\mathbb{R}^{n-1}} |u_,(x', x_n)|^p \, dx' \;\leq\; C\left(\int_{\mathbb{R}^{n-1}} |u_m(x', 0)|^p \, dx' \;+\; x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du_m(x', t)|^p \, dx' dt\right).$$

Letting $m \to \infty$ and using the fact that the first term on the right is $\|Tu_m\|_p \to \|Tu\|_p = 0$, we get the result.

Lemma. $C_c^\infty(\mathbb{R}^n)$ *is dense in* $W^{k,p}(\mathbb{R}^n)$ *for* $1 \leq k < \infty$. *In other words,* $W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$.

Proof. To prove this we choose non-negative functions $\eta, \zeta \in C_c^\infty(\mathbb{R}^n)$ such that $\eta > 0$ and for some $\delta > 0$ we have $\zeta = 1$ on $B(0, \delta)$, and $\int_{\mathbb{R}^n} \eta \, dx = 1$. Set

$$\eta_\varepsilon(x) \;=\; \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right),$$

and

$$\zeta_\varepsilon(x) \;=\; \zeta(\varepsilon x).$$

For $u \in W^{k,p}(\mathbb{R}^n)$, by previous results we have that $\eta_\varepsilon * u \to u$ in $W^{k,p}(\mathbb{R}^n)$ as $\varepsilon \to 0$, so $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$. Now for $u \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$, we claim that $\zeta_\varepsilon \cdot u \to u$ in $W^{k,p}(\mathbb{R}^n)$ as $\varepsilon \to 0$, which will prove the result. To see this, it is clear from the dominated convergence theorem that $\zeta_\varepsilon \cdot u \to u$ in $L^p(\mathbb{R}^n)$. If $k \geq 1$, we have

$$D(\zeta_\varepsilon \cdot u)(x) \;=\; \varepsilon (D\zeta)(\varepsilon x) u(x) \;+\; \zeta(\varepsilon x) Du(x).$$

The second term converges to Du in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$, and the first term converges to zero in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$. This shows that $\zeta_\varepsilon \cdot u$ converges to u in $W^{1,p}(\mathbb{R}^n)$. The case for k larger is similar, using the Leibnitz rule to compute $D^\alpha(\zeta_\varepsilon \cdot u)$.

Theorem. *(The simplest Sobolev Inequality). If* $u \in W^{n,1}(\mathbb{R}^n)$ *then* $u \in C_0(\mathbb{R}^n)$ *and*

$$(*) \qquad \|u\|_{L^\infty(\mathbb{R}^n)} \ \leq \ \|D^n u\|_{L^1(\mathbb{R}^n)}.$$

Proof. By the usual approximation arguments we just need to prove (*) for $u \in C_c^\infty(\mathbb{R}^n)$. However, we can write

$$u(x) \ = \ \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} \partial_{x_1}\partial_{x_2} \ldots \partial_{x_n} u(y_1, y_2, \ldots, y_n)\, dy_1 dy_2 \ldots dy_n.$$

Hence

$$|u(x)| \ \leq \ \|\partial_{x_1}\partial_{x_2} \ldots \partial_{x_n} u\|_{L^1}.$$

It is also clear that u is continuous.

Gagliardo-Nirenberg-Sobolev Inequaltiy. *If* $1 \leq p < n$ *and* $u \in C_c^1(\mathbb{R}^n)$ *then*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \ \leq \ C\|Du\|_{L^p(\mathbb{R}^n)},$$

where p^* *is defined by*

$$\frac{1}{p^*} \ = \ \frac{1}{p} \ - \ \frac{1}{n}.$$

Remarks. 1. What is the relationship between this result and the simple Sobolev inequality from last time? Can we iterate this inequality to get the other one? No, but let's try. Let us look at what happens if we start from $p = 1$ and form the sequence $p, p^*, p^2 = p^{**}, \ldots$. We get

$$1, \ \frac{n}{n-1}, \ \frac{n}{n-2}, \ \frac{n}{n-3}, \ \ldots, \ \frac{n}{1}, \ \infty.$$

From G-N-S we get

$$\|Du\|_{L^n(\mathbb{R}^n)} \ \leq \ C_2\|D^2u\|_{L^{n/2}(\mathbb{R}^n)} \ \leq \ C_3\|D^3u\|_{L^{n/3}(\mathbb{R}^n)} \ \leq \ \ldots \ \leq \ C_n\|D^nu\|_{L^1(\mathbb{R}^n)}.$$

However, G-N-S does not give

$$\|u\|_{L^\infty(\mathbb{R}^n)} \ \leq \ \|Du\|_{L^n(\mathbb{R}^n)}.$$

Indeed, this inequality is false.

2. How can we see that p^* should be the correct exponent?

$$u_\lambda(x) = u(\lambda x),$$

then

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \ = \ \lambda^{-n/q}\|u\|_{L^q(\mathbb{R}^n)},$$

but

$$\|Du_\lambda\|_{L^p(\mathbb{R}^n)} \;=\; \lambda^{1-n/p}\|Du\|_{L^p(\mathbb{R}^n)}.$$

Hence if

$$\|u_\alpha\|_{L^{p^*}(\mathbb{R}^n)} \;\le\; C\|Du_\alpha\|_{L^p(\mathbb{R}^n)}$$

holds for all λ we must have $-n/q = 1 - n/p$. Hence $q = p^*$.

Proof of G-N-S for $n = 3$. We will need H'"older's inequality in the form

$$\int |f|^{1/2}|g|^{1/2} \;\le\; \left(\int |f|\right)^{1/2}\left(\int |g|\right)^{1/2}.$$

For $u \in C^1_c(\mathbb{R}^n)$ we have

$$u(x_1,x_2,x_3) \;=\; \int_{-\infty}^{x_1} \partial_{x_1} u(y_1,x_2,x_3)\,dy_1,$$

and hence

$$|u(x_1,x_2,x_3)| \;\le\; \int_{\mathbb{R}} |Du|(y_1,x_2,x_3)\,dy_1.$$

Then interchanging the roles of the variables we get

$$|u(x)|^{3/2} \;\le\; \left(\int_{\mathbb{R}} |Du|(y_1,x_2,x_3)\,dy_1\right)^{1/2}\left(\int_{\mathbb{R}} |Du|(x_1,y_2,x_3)\,dy_2\right)^{1/2}\left(\int_{\mathbb{R}} |Du|(x_1,x_2,y_3)\,dy_3\right)^{1/2}.$$

Integrating with respect to x_1 and applying Hölder's inequality, we get

$$\int_{\mathbb{R}} |u(x)|^{3/2}\,dx_1 \;\le\; \left(\int_{\mathbb{R}} |Du|(y_1,x_2,x_3)\,dy_1\right)^{1/2}$$
$$\times\;\left(\iint_{\mathbb{R}^2} |Du|(x_1,y_2,x_3)\,dx_1dy_2\right)^{1/2}\left(\iint_{\mathbb{R}^2} |Du|(x_1,x_2,y_3)\,dx_1dy_3\right)^{1/2}.$$

Integrating with respect to x_2 and applying Hölder's inequality, we get

$$\iint_{\mathbb{R}^2} |u(x)|^{3/2}\,dx_1dx_2 \;\le\; \left(\iint_{\mathbb{R}} |Du|(y_1,x_2,x_3)\,dy_1dx_2\right)^{1/2}$$
$$\times\;\left(\iint_{\mathbb{R}^2} |Du|(x_1,y_2,x_3)\,dx_1dy_2\right)^{1/2}\left(\iiint_{\mathbb{R}^3} |Du|(x_1,x_2,y_3)\,dx_1dx_2dy_3\right)^{1/2}.$$

Integrating with respect to x_3 and applying Hölder's inequality, we get

$$\iiint_{\mathbb{R}^3} |u(x)|^{3/2}\,dx \;\le\; \left(\iiint_{\mathbb{R}} |Du|(y_1,x_2,x_3)\,dy_1dx_2dx_3\right)^{1/2}$$
$$\times\;\left(\iiint_{\mathbb{R}^2} |Du|(x_1,y_2,x_3)\,dx_1dy_2dx_3\right)^{1/2}\left(\iiint_{\mathbb{R}^3} |Du|(x_1,x_2,y_3)\,dx_1dx_2dy_3\right)^{1/2}$$
$$=\;\left(\iiint_{\mathbb{R}^3} |Du|(x)\,dx\right)^{3/2}.$$

For the general case we use

$$\int |f_1|^{1/n-1}\ldots|f_{n-1}|^{1/n-1} \;\le\; \left(\int |f_1|\right)^{1/n-1}\ldots\left(\int |f_{n-1}|\right)^{1/n-1}.$$